8 Matrices

INTRODUCTION

Matrices have their origin to various linear problems, the most important of which consists in the nature of solution of systems of simultaneous linear equations. In 1857, Arthur Cayley developed the properties of matrices as a pure algebraic structure. These days, matrices are used in a large number of disciplines such as Algebra, Geometry, Statistics, Physics, Chemistry and Economics etc. In this chapter, we shall learn about various types of matrices and the basic operations on matrices.

8.1 MATRIX

A rectangular arrangement of numbers, in the form of horizontal and vertical lines, is called a ***matrix.***

Horizontal lines are called ***rows*** and vertical lines are called ***columns***. Each number of a matrix is called its ***element***. The elements of a matrix are enclosed in brackets [ ].

Order of a matrix. If a matrix contains m rows and n columns, then it is called a matrix of ***order*** $m \times n$ (read as m by n).

A matrix of order $m \times n$ has mn elements.

An element appearing in the ith row and jth column of a matrix is called its *(i, j)th element* or the *(i, j)th entry.*

Notation. Matrices are usually denoted by capital letters, and the elements of a matrix by a small letter of the alphabet alongwith two suffixes, the first one indicating the number of row and the latter one, the number of the column in which the element appears. Thus, a matrix of order $m \times n$ may be written as

$$A = [a_{ij}]_{m \times n}$$

For example:

(1) $\begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix}$ is a matrix of order 2×2. It has 4 elements. Here (1, 1)th element = 2, (1,2)th element = 3, (2, 1)th element = 5 and (2, 2)th element = −7.

(2) $A = \begin{bmatrix} -1 & 3 & 4 \\ 2 & -5 & 11 \end{bmatrix}$ is matrix of order 2×3. It has 6 elements. Here $a_{22} = -5$, $a_{23} = 11$, $a_{12} = 3$ etc.

(3) $A = [2 \quad -1 \quad 0 \quad 6]$ is a matrix of order 1×4. It has four elements.

(4) $A = \begin{bmatrix} 2 & 9 \\ 5 & 8 \\ -1 & 0 \end{bmatrix}$ is a matrix of order 3×2. It has 6 elements.

8.1.1 Equal matrices

Two matrices A and B are called ***equal,*** *written as A = B, if and only if*

(i) A and B are of the same order i.e. number of rows in A = number of rows in B and number of columns in A = number of columns in B, and

(ii) their corresponding elements are equal i.e. the entries of A and B in the same position are equal.

Otherwise, the matrices are said to be ***unequal***, and we write A ≠ B.

For example:

(1) The matrices $A = \begin{bmatrix} 2 & 5 \\ 7 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 \\ 7 & -4 \end{bmatrix}$ are equal, because both are of the same order 2 × 2 and their corresponding entries are equal.

(2) The matrices $A = \begin{bmatrix} 2 & 3 & 0 \\ 7 & -6 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 3 \\ 7 & -6 & 5 \end{bmatrix}$ are not equal, because (1, 2)th entry of A ≠ (1, 2)th entry of B, even though both matrices A and B are of the same order 2 × 3.

(3) The matrices $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$ are equal if and only if $x = 2$ and $y = -5$.

8.1.2 Some special types of matrices

1. **Row Matrix.** *A matrix having only one row is called a* ***row matrix****.*

   For example, [1 −2 7 6] is a row matrix of order 1 × 4.

2. **Column Matrix.** *A matrix having only one column is called a* ***column matrix****.*

   For example, $\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$ is a column matrix of order 3 × 1.

3. **Square Matrix.** *A matrix in which the number of rows equals the number of columns is called a* ***square matrix****.* Thus, a matrix of order $n \times n$ is called a square matrix of order n.

   For example, $\begin{bmatrix} 2 & 0 \\ -1 & -5 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$ are square matrices of orders 2 and 3 respectively.

   In a square matrix, the diagonal from the left top to the right bottom is called **principal** (or **leading**) **diagonal,** and all the elements in it are called **diagonal elements.**

   In the square matrix $\begin{bmatrix} 2 & 0 \\ -1 & -5 \end{bmatrix}$, the principal diagonal consists of 2, −5.

4. **Zero (or Null) Matrix.** *A matrix whose each element is zero is called a* ***zero*** *(or* ***null****)* ***matrix****.*

   For example, $[\,0 \;\; 0\,]$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are all zero matrices of orders 1 × 2, 2 × 1, 2 × 2 and 2 × 3 respectively.

5. **Identity (or Unit) Matrix.** *A square matrix in which each diagonal element is 1 and all other elements are zero is called an **identity** (or **unit**) **matrix**.*

For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of orders 2 and 3 respectively.

Illustrative Examples

Example 1. *If a matrix has 6 elements, what are the possible orders it can have?*

Solution. Since all matrices of order 1×6, 6×1, 2×3 or 3×2 contain 6 elements, a matrix containing 6 elements can have any one of the following orders:

$$1 \times 6,\ 6 \times 1,\ 2 \times 3 \ \text{ or } \ 3 \times 2.$$

Example 2. *Construct a 2×2 matrix whose elements a_{ij} are given by $a_{ij} = i + j$.*

Solution. Given $a_{ij} = i + j$,

$\therefore \quad a_{11} = 1 + 1 = 2,\ a_{12} = 1 + 2 = 3,$

$\quad a_{21} = 2 + 1 = 3,\ a_{22} = 2 + 2 = 4.$

Hence, the required matrix $= \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$.

Example 3. *Find the values of a, b, c and d if* $\begin{bmatrix} a & -2 \\ b & 7 \end{bmatrix} = \begin{bmatrix} 2 & c \\ 3 & 2c + d \end{bmatrix}$.

Solution. Given $\begin{bmatrix} a & -2 \\ b & 7 \end{bmatrix} = \begin{bmatrix} 2 & c \\ 3 & 2c + d \end{bmatrix}$.

By definition of equality of matrices, we get

$\quad a = 2,\ b = 3,\ c = -2 \ \text{ and } \ 2c + d = 7$

$\Rightarrow \quad 2 \times (-2) + d = 7 \ \Rightarrow \ d = 7 + 4 = 11.$

Hence, $a = 2,\ b = 3,\ c = -2$ and $d = 11$.

Example 4. *If* $\begin{bmatrix} x + 3y & y \\ 7 - x & 4 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 0 & 4 \end{bmatrix}$, *find the values of x and y.*

Solution. Given $\begin{bmatrix} x + 3y & y \\ 7 - x & 4 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 0 & 4 \end{bmatrix}$.

By definition of equality of matrices, we get

$\quad x + 3y = 4,\ y = -1 \ \text{ and } \ 7 - x = 0$

$\Rightarrow \quad x + 3y = 4,\ y = -1 \ \text{ and } \ x = 7.$

Note that $x = 7$ and $y = -1$ satisfy $x + 3y = 4$.

Hence, $x = 7,\ y = -1$.

Example 5. *Find the values of x, y, a and b if* $\begin{bmatrix} x + y & a + b \\ a - b & 2x - 3y \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 3 & -5 \end{bmatrix}$.

Solution. By definition of equality of matrices, we get

$x + y = 5$ ...(*i*) $\quad a + b = -1$ ...(*ii*)

$a - b = 3$ ...(*iii*) $\quad 2x - 3y = -5$ ...(*iv*)

Adding (*ii*) and (*iii*), we get

$$2a = 2 \ \Rightarrow \ a = 1.$$

Putting $a = 1$ in (*ii*), we get

$$1 + b = -1 \quad \Rightarrow \quad b = -2.$$

To find x and y, multiplying (*i*) by 3, we get

$$3x + 3y = 15 \qquad \ldots(v)$$

Adding (*iv*) and (*v*), we get

$$5x = 10 \quad \Rightarrow \quad x = 2.$$

Putting $x = 2$ in (*i*), we get

$$2 + y = 5 \quad \Rightarrow \quad y = 5 - 2 = 3.$$

Hence, $x = 2, y = 3, a = 1$ and $b = -2$.

Example 6. *If* $\begin{bmatrix} 2x+2 & y^2+2 \\ 5 & y^2-5y \end{bmatrix} = \begin{bmatrix} x+4 & 3y \\ 5 & -6 \end{bmatrix}$, *find the values of x and y.*

Solution. Given $\begin{bmatrix} 2x+2 & y^2+2 \\ 5 & y^2-5y \end{bmatrix} = \begin{bmatrix} x+4 & 3y \\ 5 & -6 \end{bmatrix}$

By definition of equality of matrices, we get

$$2x + 2 = x + 4 \qquad \ldots (i)$$

$$y^2 + 2 = 3y \qquad \ldots (ii)$$

$$y^2 - 5y = -6 \qquad \ldots (iii)$$

From (*i*), we get

$$2x - x = 4 - 2 \quad \Rightarrow \quad x = 2.$$

From (*ii*), we get

$$y^2 + 2 = 3y \quad \Rightarrow \quad y^2 - 3y + 2 = 0$$

$$\Rightarrow \quad (y-1)(y-2) = 0 \quad \Rightarrow \quad y - 1 = 0 \text{ or } y - 2 = 0$$

$$\Rightarrow \quad y = 1, 2.$$

From (*iii*), we get

$$y^2 - 5y = -6 \quad \Rightarrow \quad y^2 - 5y + 6 = 0$$

$$\Rightarrow \quad (y-2)(y-3) = 0 \quad \Rightarrow \quad y - 2 = 0 \text{ or } y - 3 = 0$$

$$\Rightarrow \quad y = 2, 3.$$

Since (*ii*) and (*iii*) must be true simultaneously, we take the common value of y. Therefore, $y = 2$.

Hence, $x = 2$ and $y = 2$.

Exercise 8.1

1. Classify the following matrices :

(*i*) $\begin{bmatrix} 2 & -1 \\ 5 & 1 \end{bmatrix}$ (*ii*) $[2 \quad 3 \quad -7]$ (*iii*) $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$

(*iv*) $\begin{bmatrix} 2 & -4 \\ 0 & 0 \\ 1 & 7 \end{bmatrix}$ (*v*) $\begin{bmatrix} 2 & 7 & 8 \\ -1 & \sqrt{2} & 0 \end{bmatrix}$ (*vi*) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

2. (*i*) If a matrix has 4 elements, what are the possible orders it can have?

(*ii*) If a matrix has 8 elements, what are the possible orders it can have?

3. Construct a 2×2 matrix whose elements a_{ij} are given by

(*i*) $a_{ij} = 2i - j$ (*ii*) $a_{ij} = i.j$