Ratio and Proportion

7

INTRODUCTION

In previous classes, you have already read by ratio and proportion. In this chapter, we shall review ratio and will learn more about proportion. We shall also learn about continued proportion, mean proportion, componendo, dividendo, alternendo, invertendo and their combinations, some applications on proportion.

7.1 RATIO

A **ratio** is a comparison of the sizes of two or more quantities of the same kind by division.

If a and b are two quantities of the same kind (in same units), then the fraction $\frac{a}{b}$ *is called the* ***ratio*** *of a to b.* It is written as $a : b$ (read 'a is to b'). Thus, the ratio of a to $b = \frac{a}{b}$ or $a : b$.

For example, the ratio of ₹8 to ₹13 is $\frac{8}{13}$ or 8 : 13.

The quantities a and b are called the ***terms*** of the ratio; a is called the ***first term*** (or ***antecedent***) and b is called the ***second term*** (or ***consequent***).

Some facts about ratio :

- Since ratio is a fraction, both of its terms can be multiplied or divided by the same (non-zero) number.

  For example: $10 : 15 = \frac{10}{15} = \frac{10 \times 2}{15 \times 2} = \frac{20}{30} \Rightarrow 10 : 15 = 20 : 30.$

  Also $10 : 15 = \frac{10}{15} = \frac{10 \div 5}{15 \div 5} = \frac{2}{3} \Rightarrow 10 : 15 = 2 : 3.$

  Usually, the ratio is expressed in its lowest terms.
- The order of the terms in a ratio is important.
- Ratio is taken only between positive quantities.
- Ratio exists only between two quantities of the same kind. For example, there is no ratio between ₹20 and 13 m.
- While finding the ratio between two quantities, both the quantities must be in the same units. For example, to find ratio of 35 paise to ₹1·75, first express ₹1·75 in paise *i.e.* ₹1·75 = (1·75 × 100) paise = 175 paise,

  ∴ ratio of 35 paise to ₹1·75 = $\frac{35}{175} = \frac{1}{5}$ or 1 : 5.
- A ratio is a number, so it has no units.

- If the terms of a given ratio are fractions, then convert them in whole numbers by multiplying each term by the L.C.M. of their denominators. For example,

  $\frac{1}{2} : \frac{1}{3} = \frac{1}{2} \times 6 : \frac{1}{3} \times 6 = 3 : 2.$

- *To compare two ratios*, convert them into equivalent like fractions.
- *Increase or decrease in a ratio.*

  If a quantity increases or decreases in the ratio $a : b$, then new quantity $= \frac{b}{a}$ of the original quantity.

For example:

Let the price of an article increases from ₹8 to ₹10, we say that the price has increased in the ratio 8 : 10 *i.e.* 4 : 5. Thus,

$$\text{new price} = \frac{5}{4} \text{ of the original price.}$$

Similarly, if the price of an article decreases from ₹10 to ₹8, we say that the price has decreased in the ratio 10 : 8 *i.e.* 5 : 4. Thus,

$$\text{new price} = \frac{4}{5} \text{ of the original price.}$$

The fraction by which an original quantity is multiplied to get a new quantity is called ***multiplying factor.***

Ratio a : b : c

Three quantities of the same kind (in same units) are said to be in the ratio a : b : c if the quantities are ak, bk and ck respectively, where k is any non-zero real number.

Similarly, four quantities of the same kind (in same units) are said to be in the ratio $a : b : c : d$ if the quantities are ak, bk, ck and dk respectively, where k is any non-zero real number.

7.1.1 Composition of ratios

1. *When two or more ratios are multiplied together, they are said to be* ***compounded****.* Thus, if $\frac{a}{b}$ and $\frac{c}{d}$ are any two ratios, then $\frac{ac}{bd}$ is their compounded ratio.

$\therefore$ ***Compounded ratio of a : b and c : d is ac : bd.***

Similarly, **compounded ratio of** $a : b$, $c : d$ **and** $e : f$ **is** $ace : bdf$.

2. *A ratio compounded with itself is called* ***duplicate*** *ratio of the given ratio.*

$\therefore$ ***duplicate ratio of a : b is*** $a^2 : b^2$.

Similarly, ***triplicate ratio of a : b is*** $a^3 : b^3$.

Sub-duplicate ratio of a : b is $\sqrt{a} : \sqrt{b}$.

Sub-triplicate ratio of a : b is $\sqrt[3]{a} : \sqrt[3]{b}$.

3. The reciprocal ratio of $a : b$ **is** $b : a$.

Illustrative Examples

Example 1. *Find the compounded ratio of :*

(i) 5 : 7 and 9 : 10

(ii) 2 : 3, 4 : 5 and 6 : 7

(iii) $(x + y) : (x - y)$, $(x^2 + y^2) : (x + y)^2$ and $(x^2 - y^2)^2 : (x^4 - y^4)$.

Solution.

(i) Required ratio $= \frac{5}{7} \times \frac{9}{10} = \frac{9}{14} = 9 : 14.$

(ii) Required ratio $= \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} = \frac{16}{35} = 16 : 35.$

(iii) Required ratio $= \frac{x+y}{x-y} \times \frac{x^2+y^2}{(x+y)^2} \times \frac{(x^2-y^2)^2}{x^4-y^4}$

$$= \frac{x^2+y^2}{(x-y)(x+y)} \times \frac{(x^2-y^2)^2}{(x^2-y^2)(x^2+y^2)}$$

$$= \frac{x^2-y^2}{x^2-y^2} = \frac{1}{1} = 1 : 1.$$

Example 2. *Find the following :*

(i) the duplicate ratio of 3 : 7. *(ii) the triplicate ratio of 2 : 5.*

(iii) the sub-duplicate ratio of 36 : 25. *(iv) the sub-triplicate ratio of 216 : 125.*

(v) the reciprocal ratio of 9 : 11.

Solution.

(i) The duplicate ratio of 3 : 7 is $3^2 : 7^2$ *i.e.* 9 : 49.

(ii) The triplicate ratio of 2 : 5 is $2^3 : 5^3$ *i.e.* 8 : 125.

(iii) The sub-duplicate ratio of 36 : 25 is $\sqrt{36} : \sqrt{25}$ *i.e.* 6 : 5.

(iv) The sub-triplicate ratio of 216 : 125 is $\sqrt[3]{216} : \sqrt[3]{125}$ *i.e.* 6 : 5.

(v) The reciprocal ratio of 9 : 11 is 11 : 9.

Example 3. *Arrange the following ratios in descending order of magnitude*

2 : 3, 8 : 15, 11 : 12 and 7 : 16.

Solution. Given ratios are $\frac{2}{3}, \frac{8}{15}, \frac{11}{12}$ and $\frac{7}{16}$.

We convert them into equivalent like fractions.

L.C.M. of 3, 15, 12, 16 = 240

$$\frac{2}{3} = \frac{2 \times 80}{3 \times 80} = \frac{160}{240}, \quad \frac{8}{15} = \frac{8 \times 16}{15 \times 16} = \frac{128}{240},$$

$$\frac{11}{12} = \frac{11 \times 20}{12 \times 20} = \frac{220}{240}, \quad \frac{7}{16} = \frac{7 \times 15}{16 \times 15} = \frac{105}{240}.$$

As 220 > 160 > 128 > 105, $\frac{220}{240} > \frac{160}{240} > \frac{128}{240} > \frac{105}{240}$

$\Rightarrow \quad \frac{11}{12} > \frac{2}{3} > \frac{8}{15} > \frac{7}{16}$

Hence, the given ratios in descending order of magnitude are

11 : 12, 2 : 3, 8 : 15 and 7 : 16.

Example 4. *If A : B = 6 : 7 and B : C = 8 : 9, find A : B : C.*

Solution. Given A : B = 6 : 7 and B : C = 8 : 9.

L.C.M. of two values of B

i.e. 7 and 8 is 56.

> To find A : B : C, make B same in both cases

Thus, A : B = 6 : 7 = 48 : 56 and

B : C = 8 : 9 = 56 : 63.

$\therefore$ A : B : C = 48 : 56 : 63, which is in simplest form.

Example 5. *If 3A = 5B = 6C, find A : B : C.*

Solution. Let $3A = 5B = 6C = x$, then

$$A = \frac{x}{3},\ B = \frac{x}{5},\ C = \frac{x}{6}.$$

$$\therefore \quad A : B : C = \frac{x}{3} : \frac{x}{5} : \frac{x}{6} = \frac{1}{3} : \frac{1}{5} : \frac{1}{6}$$ (Convert ratio in whole numbers)

$$= \frac{1}{3} \times 30 : \frac{1}{5} \times 30 : \frac{1}{6} \times 30$$

L.C.M. of 3, 5, 6 = 30

$$= 10 : 6 : 5.$$

Example 6. *If (3a + 2b) : (5a + 3b) = 18 : 29, find a : b.*

Solution. Given $(3a + 2b) : (5a + 3b) : 18 : 29$

$\Rightarrow \quad \frac{3a + 2b}{5a + 3b} = \frac{18}{29}$

$\Rightarrow \quad 90a + 54b = 87a + 58\,b$

$\Rightarrow \quad 3a = 4b \qquad \Rightarrow \frac{a}{b} = \frac{4}{3}.$

Hence, $a : b = 4 : 3$.

Example 7. *If a : b = 5 : 3, find (5a + 8b) : (6a – 7b).*

Solution. Given $a : b = 5 : 3 \Rightarrow \frac{a}{b} = \frac{5}{3}$ ...(*i*)

$$\therefore \quad \frac{5a + 8b}{6a - 7b} = \frac{5.\frac{a}{b} + 8}{6.\frac{a}{b} - 7}$$ (Dividing num. and denom. by *b*)

$$= \frac{5.\frac{5}{3} + 8}{6.\frac{5}{3} - 7}$$ (Using (*i*))

$$= \frac{\frac{25}{3} + 8}{10 - 7} = \frac{\frac{25 + 24}{3}}{3} = \frac{49}{3} \times \frac{1}{3} = \frac{49}{9}.$$

Hence, $(5a + 8b) : (6a - 7b) = 49 : 9$.

Example 8. *If $(2x^2 - 5y^2) : xy = 1 : 3$, find the ratio x : y.*

Solution. Given $(2x^2 - 5y^2) : xy = 1 : 3 \Rightarrow \frac{2x^2 - 5y^2}{xy} = \frac{1}{3}$

$\Rightarrow \quad 6x^2 - 15y^2 = xy$

$\Rightarrow \quad 6x^2 - xy - 15y^2 = 0$

$\Rightarrow \quad (3x - 5y)(2x + 3y) = 0$

$\Rightarrow \quad 3x - 5y = 0$ or $2x + 3y = 0$

$\Rightarrow \quad 3x = 5y$ or $2x = -3y$

$\Rightarrow \quad \frac{x}{y} = \frac{5}{3}$ or $\frac{x}{y} = -\frac{3}{2}.$

But $x : y$ is not defined for negative values of x or y.
Hence, $x : y = 5 : 3$.

Example 9. *If (5a – 2b) : (2a + b) = (6a – b) : (8a – b), find a : b.*

Solution. Given $(5a - 2b) : (2a + b) = (6a - b) : (8a - b)$

$\Rightarrow \quad \dfrac{5a-2b}{2a+b} = \dfrac{6a-b}{8a-b}$

$\Rightarrow \quad (5a - 2b)(8a - b) = (2a + b)(6a - b)$

$\Rightarrow \quad 40a^2 - 5ab - 16ab + 2b^2 = 12a^2 - 2ab + 6ab - b^2$

$\Rightarrow \quad 40a^2 - 21ab + 2b^2 = 12a^2 + 4ab - b^2$

$\Rightarrow \quad 28a^2 - 25ab + 3b^2 = 0$

$\Rightarrow \quad (4a - 3b)(7a - b) = 0$

$\Rightarrow \quad 4a - 3b = 0$ or $7a - b = 0$

$\Rightarrow \quad 4a = 3b$ or $7a = b$

$\Rightarrow \quad \dfrac{a}{b} = \dfrac{3}{4}$ or $\dfrac{a}{b} = \dfrac{1}{7}$.

Hence, $a : b = 3 : 4$ or $a : b = 1 : 7$.

Example 10. *Two numbers are in the ratio 7 : 11. If 7 is added to each of the numbers, the ratio becomes 2 : 3. Find the numbers.*

Solution. Since the numbers are in the ratio 7 : 11, let the required numbers be $7x$ and $11x$.

According to given, $(7x + 7) : (11x + 7) = 2 : 3$

$\Rightarrow \quad \dfrac{7x+7}{11x+7} = \dfrac{2}{3}$

$\Rightarrow \quad 22x + 14 = 21x + 21 \Rightarrow x = 7$

$\Rightarrow \quad 7x = 7 \times 7 = 49$ and $11x = 11 \times 7 = 77$.

$\therefore$ The required numbers are 49 and 77.

Example 11. *A sum of money is divided in the ratio 3 : 5. If the larger part is ₹3125, find the smaller part.*

Solution. Let the smaller part of money be ₹x.

According to given, $\dfrac{x}{3125} = \dfrac{3}{5} \quad \Rightarrow \quad x = \dfrac{3}{5} \times 3125 = 1875$.

$\therefore$ The smaller part of the money = ₹1875.

Example 12. *Divide ₹581 among three children in the ratio $1\frac{1}{4} : 1\frac{1}{3} : \frac{7}{8}$.*

Solution. Given ratio $= 1\dfrac{1}{4} : 1\dfrac{1}{3} : \dfrac{7}{8}$

$= \dfrac{5}{4} : \dfrac{4}{3} : \dfrac{7}{8}$ (Convert ratio in the whole numbers)

$= \dfrac{5}{4} \times 24 : \dfrac{4}{3} \times 24 : \dfrac{7}{8} \times 24$ L.C.M. of 4, 3, 8 = 24

$= 30 : 32 : 21$.

Sum of the terms of the ratio $= 30 + 32 + 21 = 83$.

Share of first child $= \dfrac{30}{83}$ of ₹581 = ₹$\left(\dfrac{30}{83} \times 581\right)$ = ₹210,

share of second child $= \dfrac{32}{83}$ of ₹581 = ₹$\left(\dfrac{32}{83} \times 581\right)$ = ₹224,

share of third child $= \dfrac{21}{83}$ of ₹581 = ₹$\left(\dfrac{21}{83} \times 581\right)$ = ₹147.

Example 13. *Divide ₹1870 into three parts in such a way that half of the first part, one-third of the second part and one-sixth of the third part are all equal.*

Solution. Let $\frac{1}{2}$ (1st part) = $\frac{1}{3}$ (2nd part) = $\frac{1}{6}$ (3rd part) = x

$\Rightarrow$ 1st part = $2x$, 2nd part = $3x$, 3rd part = $6x$.

According to given, $2x + 3x + 6x = 1870$

$\Rightarrow$ $11x = 1870 \Rightarrow x = 170$.

$\therefore$ 1st part = ₹(2 × 170) = ₹340,

2nd part = ₹(3 × 170) = ₹510 and

3rd part = ₹(6 × 170) = ₹1020.

Example 14. *A bag contains ₹155 in the form of 1-rupee, 50-paise and 10-paise coins in the ratio of 3 : 5 : 7. Find the total number of coins.*

Solution. Since 1-rupee, 50-paise and 10-paise coins are in the ratio 3 : 5 : 7, let the number of these coins be $3x$, $5x$ and $7x$ respectively.

$\therefore$ The value of 1-rupee coins = ₹$(3x \times 1)$ = ₹$3x$,

the value of 50-paise coins = ₹$\left(5x \times \frac{1}{2}\right)$ = ₹$\frac{5x}{2}$,

the value of 10-paise coins = ₹$\left(7x \times \frac{1}{10}\right)$ = ₹$\frac{7x}{10}$.

According to given, $3x + \frac{5}{2}x + \frac{7}{10}x = 155$

$\Rightarrow$ $30x + 25x + 7x = 155 \times 10$

$\Rightarrow$ $62x = 1550 \Rightarrow x = 25$.

$\therefore$ The number of 1-rupee coins = 3 × 25 = 75,

the number of 50-paise coins = 5 × 25 = 125 and

the number of 10-paise coins = 7 × 25 = 175.

$\therefore$ The total number of coins = 75 + 125 + 175 = 375.

Example 15. *There are 36 members on a student council in a school and the ratio of the number of boys to the number of girls is 3 : 1. How many more girls should be added to the council so that the ratio of number of boys to the number of girls may be 9 : 5?*

Solution. Sum of the terms of the ratio = 3 + 1 = 4,

$\therefore$ the number of boys in the council = $\frac{3}{4} \times 36 = 27$

and the number of girls in the council = $\frac{1}{4} \times 36 = 9$.

Let x girls be added to the student council, then the number of girls in the council = $9 + x$.

According to given, $27 : (9 + x) = 9 : 5$

$\Rightarrow$ $\frac{27}{9+x} = \frac{9}{5} \Rightarrow 9(9 + x) = 27 \times 5$

$\Rightarrow$ $9 + x = 3 \times 5 \Rightarrow x = 15 - 9$

$\Rightarrow$ $x = 6$.

Hence, the number of girls to be added to the student council = 6.

Example 16. *Work done by $(x - 3)$ men in $(2x + 1)$ days and the work done by $(2x + 1)$ men in $(x + 4)$ days are in the ratio 3 : 10. Find the value of x.*

Solution. Assuming that all men do the same amount of work in one day, we have :

The quantity of work done by $(x - 3)$ men in $(2x + 1)$ days

$= (x - 3)(2x + 1)$ times the quantity of work done by 1 man in one day.

The quantity of work done by $(2x + 1)$ men in $(x + 4)$ days

$= (2x + 1)(x + 4)$ times the quantity of work done by 1 man in one day.

According to given, $\frac{(x-3)(2x+1)}{(2x+1)(x+4)} = \frac{3}{10}$

$\Rightarrow \quad \frac{x-3}{x+4} = \frac{3}{10}$ $\quad$ [$\because$ $x = -\frac{1}{2}$ is not possible, so we cancel factor $(2x + 1)$]

$\Rightarrow \quad 10x - 30 = 3x + 12$

$\Rightarrow \quad 7x = 42 \quad \Rightarrow x = 6.$

Exercise 7.1

1. An alloy consists of $27\frac{1}{2}$ kg of copper and $2\frac{3}{4}$ kg of tin. Find the ratio by weight of tin to the alloy.
2. Find the compounded ratio of :
   (*i*) $2 : 3$ and $4 : 9$ (*ii*) $4 : 5$, $5 : 7$ and $9 : 11$
   (*iii*) $(a - b) : (a + b)$, $(a + b)^2 : (a^2 + b^2)$ and $(a^4 - b^4) : (a^2 - b^2)^2$.
3. Find the duplicate ratio of :
   (*i*) $2 : 3$ (*ii*) $\sqrt{5} : 7$ (*iii*) $5a : 6b$.
4. Find the triplicate ratio of :
   (*i*) $3 : 4$ (*ii*) $\frac{1}{2} : \frac{1}{3}$ (*iii*) $1^3 : 2^3$.
5. Find the sub-duplicate ratio of :
   (*i*) $9 : 16$ (*ii*) $\frac{1}{4} : \frac{1}{9}$ (*iii*) $9a^2 : 49b^2$.
6. Find the sub-triplicate ratio of :
   (*i*) $1 : 216$ (*ii*) $\frac{1}{8} : \frac{1}{125}$ (*iii*) $27a^3 : 64b^3$.
7. Find the reciprocal ratio of :
   (*i*) $4 : 7$ (*ii*) $3^2 : 4^2$ (*iii*) $\frac{1}{9} : 2$.
8. Arrange the following ratios in ascending order of magnitude:
   $2 : 3$, $17 : 21$, $11 : 14$ and $5 : 7$.
9. (*i*) If $A : B = 2 : 3$, $B : C = 4 : 5$ and $C : D = 6 : 7$, find $A : D$.
   (*ii*) If $x : y = 2 : 3$ and $y : z = 4 : 7$, find $x : y : z$.
10. (*i*) If $A : B = \frac{1}{4} : \frac{1}{5}$ and $B : C = \frac{1}{7} : \frac{1}{6}$, find $A : B : C$.
    (*ii*) If $3A = 4B = 6C$, find $A : B : C$.
11. (*i*) If $\frac{3x + 5y}{3x - 5y} = \frac{7}{3}$, find $x : y$.
    (*ii*) If $a : b = 3 : 11$, find $(15a - 3b) : (9a + 5b)$.
12. (*i*) If $(4x^2 + xy) : (3xy - y^2) = 12 : 5$, find $(x + 2y) : (2x + y)$.
    (*ii*) If $y(3x - y) : x(4x + y) = 5 : 12$, find $(x^2 + y^2) : (x + y)^2$.