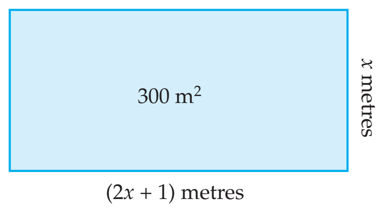
5 Quadratic Equations in One Variable

INTRODUCTION

In previous classes, we have learnt that an algebraic expression of the form $ax^2 + bx + c$, where a, b, c are real numbers and $a \neq 0$, is called a quadratic polynomial in the variable x. When this expression is equated to zero, we get a quadratic equation. Thus, $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$, is called a quadratic equation in the variable x.

In real life, we come across many situations when we get quadratic equations. For example, to build a prayer hall of carpet area 300 m^2 with its length 1 metre more than twice its breadth, what should be the dimensions of the hall?

If the breadth of the hall is x metres then its length is $(2x + 1)$ metres. This information pictorially is shown in the adjoining figure.



Area of hall $=$ length $\times$ breadth

$= (2x + 1)\, x$ m^2

$= 300$ m^2 (given)

$\Rightarrow \quad (2x + 1)\, x = 300$

$\Rightarrow \quad 2x^2 + x - 300 = 0.$

Thus, the breadth of the hall satisfies the quadratic equation $2x^2 + x - 300 = 0$.

It is believed that Babylonians were the first to solve quadratic equations. Indian Mathematician, Brahmagupta (A.D. 598 – 665) gave an explicit formula to solve a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$. Later on, Sridhara Acharyya (A.D. 1025) derived a formula, known as quadratic formula for solving a quadratic equation by the method of completing the squares.

In this chapter, we shall learn various methods of finding the roots (or solutions) of quadratic equations in one variable and will discuss the nature of roots. We shall also learn some applications of quadratic equations in daily life situations.

5.1 QUADRATIC EQUATIONS IN ONE VARIABLE

An equation of the type $ax^2 + bx + c = 0$, $a \neq 0$, is called a ***quadratic equation*** *in the variable x.*

The equation $ax^2 + bx + c = 0$, $a \neq 0$, is called the **general** (or **standard**) **form.**

For example: $x^2 - 7x + 12 = 0$ and $3x^2 - 4x - 4 = 0$ are quadratic equations in the variable x.

Note.

Simplify the given equation before deciding whether it is quadratic or not.

5.1.1 Roots (or solutions) of a quadratic equation

*A number α is a **root** (or **solution**) of the quadratic equation $ax^2 + bx + c = 0$ if and only if $a\alpha^2 + b\alpha + c = 0$.*

For example :

When we substitute $x = 3$ in the quadratic equation $x^2 - 7x + 12 = 0$, we get

$(3)^2 - 7 \times 3 + 12 = 0$ *i.e.* $9 - 21 + 12 = 0$ *i.e.* $0 = 0$, which is true, therefore, 3 is a root of the quadratic equation $x^2 - 7x + 12 = 0$.

When we substitute $x = 2$ in the quadratic equation $x^2 - 7x + 12 = 0$, we get

$(2)^2 - 7 \times 2 + 12 = 0$ *i.e.* $4 - 14 + 12 = 0$ *i.e.* $2 = 0$, which is wrong, therefore, 2 is not a root of the quadratic equation $x^2 - 7x + 12 = 0$.

Illustrative Examples

Example 1. *Check whether the following are quadratic equations:*

(i) $(x - 2)^2 + 1 = 2x - 3$ *(ii)* $(2x - 1)(x - 3) = (x + 5)(x - 1)$

(iii) $x(x + 1) + 8 = (x + 2)(x - 2)$ *(iv)* $(x + 2)^3 = x^3 - 4$

Solution. (*i*) Given $(x - 2)^2 + 1 = 2x - 3$

$\Rightarrow \quad x^2 - 4x + 4 + 1 = 2x - 3 \Rightarrow x^2 - 4x + 5 = 2x - 3$

$\Rightarrow \quad x^2 - 4x - 2x + 5 + 3 = 0$

$\Rightarrow \quad x^2 - 6x + 8 = 0$, which is of the form $ax^2 + bx + c = 0$.

Therefore, the given equation is a quadratic equation.

(*ii*) Given $(2x - 1)(x - 3) = (x + 5)(x - 1)$

$\Rightarrow \quad 2x^2 - 6x - x + 3 = x^2 - x + 5x - 5 \Rightarrow 2x^2 - 7x + 3 = x^2 + 4x - 5$

$\Rightarrow \quad 2x^2 - x^2 - 7x - 4x + 3 + 5 = 0$

$\Rightarrow \quad x^2 - 11x + 8 = 0$, which is of the form $ax^2 + bx + c = 0$.

Therefore, the given equation is a quadratic equation.

(*iii*) Given $x(x + 1) + 8 = (x + 2)(x - 2)$

$\Rightarrow \quad x^2 + x + 8 = x^2 - 4 \Rightarrow x^2 - x^2 + x + 8 + 4 = 0$

$\Rightarrow \quad x + 12 = 0$, which is not of the form $ax^2 + bx + c = 0$.

Therefore, the given equation is not a quadratic equation.

(*iv*) Given $(x + 2)^3 = x^3 - 4$

$\Rightarrow \quad x^3 + 2^3 + 3.x.2\,(x + 2) = x^3 - 4$

$\Rightarrow \quad x^3 + 8 + 6x^2 + 12x = x^3 - 4 \Rightarrow x^3 - x^3 + 6x^2 + 12x + 8 + 4 = 0$

$\Rightarrow \quad 6x^2 + 12x + 12 = 0$

$\Rightarrow \quad x^2 + 2x + 2 = 0$, which is of the form $ax^2 + bx + c = 0$.

Therefore, the given equation is a quadratic equation.

Example 2. *Check whether the following are quadratic equations:*

(i) $2x + \frac{3}{x} = 5, x \neq 0$ *(ii)* $\frac{x+2}{3} = \frac{2}{x} + x, x \neq 0.$

Solution. (*i*) Given $2x + \frac{3}{x} = 5, x \neq 0$

$\Rightarrow \quad 2x^2 + 3 = 5x$

$\Rightarrow \quad 2x^2 - 5x + 3 = 0$, which is of the form $ax^2 + bx + c = 0$.

Therefore, the given equation can be reduced to a quadratic equation.

(*ii*) Given $\frac{x+2}{3} = \frac{2}{x} + x$

$\Rightarrow \quad \frac{x+2}{3} = \frac{2+x^2}{x}$

$\Rightarrow \quad 3(2 + x^2) = x(x + 2) \;\Rightarrow\; 6 + 3x^2 = x^2 + 2x$

$\Rightarrow \quad 3x^2 - x^2 - 2x + 6 = 0 \;\Rightarrow\; 2x^2 - 2x + 6 = 0$

$\Rightarrow \quad x^2 - x + 3 = 0$, which is of the form $ax^2 + bx + c = 0$.

Therefore, the given equation can be reduced to a quadratic equation.

Example 3. *Show that $\sqrt{2}$ and $-2\sqrt{2}$ are both solutions of the quadratic equation*

$$x^2 + \sqrt{2}\,x - 4 = 0.$$

Solution. The given equation is $x^2 + \sqrt{2}\,x - 4 = 0$.

Substituting $x = \sqrt{2}$ in the LHS of the given equation, we get

LHS = $(\sqrt{2})^2 + \sqrt{2}\times\sqrt{2} - 4 = 2 + 2 - 4 = 0$ = RHS

$\Rightarrow \quad \sqrt{2}$ is a solution of the given equation.

Substituting $x = -2\sqrt{2}$ in the LHS of the given equation, we get

LHS = $(-2\sqrt{2})^2 + \sqrt{2}(-2\sqrt{2}) - 4 = 8 - 4 - 4 = 0$ = RHS

$\Rightarrow \quad -2\sqrt{2}$ is a solution of the given equation.

Hence, $\sqrt{2}$ and $-2\sqrt{2}$ are both solutions of the given quadratic equation.

Example 4. *If $\frac{2}{3}$ is a root of the equation $kx^2 - x - 2 = 0$, then find the value of k.*

Solution. Since $\frac{2}{3}$ is a root of the equation $kx^2 - x - 2 = 0$, $x = \frac{2}{3}$ satisfies the equation

i.e. $k \times \left(\frac{2}{3}\right)^2 - \frac{2}{3} - 2 = 0 \;\Rightarrow\; \frac{4}{9}k - \frac{8}{3} = 0$

$\Rightarrow \quad \frac{4}{9}k = \frac{8}{3} \;\Rightarrow\; k = \frac{8}{3}\times\frac{9}{4} \;\Rightarrow\; k = 6.$

Hence, the value of k is 6.

Example 5. *If $-a$ is a solution of the equation $x^2 + 3ax + k = 0$, find the value of k.*

Solution. Since $-a$ is a solution of the equation $x^2 + 3ax + k = 0$, $x = -a$ satisfies the given equation

i.e. $(-a)^2 + 3a(-a) + k = 0 \;\Rightarrow\; a^2 - 3a^2 + k = 0$

$\Rightarrow \quad k = 2a^2.$

Hence, the value of k is $2a^2$.

Example 6. *If 2 and 3 are roots of the equation $3x^2 - 2mx + 2n = 0$, find the values of m and n.*

Solution. The given equation is $3x^2 - 2mx + 2n = 0$.

As 2 is a root of the given equation, so $x = 2$ satisfies the given equation

i.e. $3 \times 2^2 - 2m \times 2 + 2n = 0 \;\Rightarrow\; 12 - 4m + 2n = 0$

$\Rightarrow \quad 4m - 2n - 12 = 0$ ...(*i*)

Also 3 is a root of the given equation, so $x = 3$ satisfies the given equation

i.e. $3 \times 3^2 - 2m \times 3 + 2n = 0 \;\Rightarrow\; 27 - 6m + 2n = 0$

$\Rightarrow \quad 6m - 2n - 27 = 0$ ...(*ii*)

Subtracting (*i*) from (*ii*), we get

$$2m - 15 = 0 \Rightarrow m = \frac{15}{2}.$$

Substituting $m = \frac{15}{2}$ in (*i*), we get

$$4 \times \frac{15}{2} - 2n - 12 = 0 \Rightarrow 30 - 2n - 12 = 0$$

$$\Rightarrow 18 - 2n = 0 \Rightarrow n = 9.$$

Hence, $m = \frac{15}{2}$ and $n = 9$.

Exercise 5.1

1. Check whether the following are quadratic equations:

(*i*) $\sqrt{3}\,x^2 - 2x + \frac{3}{5} = 0$ (*ii*) $(2x + 1)(3x - 2) = 6(x + 1)(x - 2)$

(*iii*) $(x - 3)^3 + 5 = x^3 + 7x^2 - 1$ (*iv*) $x - \frac{3}{x} = 2, x \neq 0$

(*v*) $x + \frac{2}{x} = x^2, x \neq 0$ (*vi*) $x^2 + \frac{1}{x^2} = 3, x \neq 0$

2. In each of the following, determine whether the given numbers are roots of the given equations or not:

(*i*) $x^2 - x + 1 = 0;\ 1, -1$ (*ii*) $x^2 - 5x + 6 = 0;\ 2, -3$

(*iii*) $3x^2 - 13x - 10 = 0;\ 5, -\frac{2}{3}$ (*iv*) $6x^2 - x - 2 = 0;\ -\frac{1}{2}, \frac{2}{3}$

3. In each of the following, determine whether the given numbers are solutions of the given equations or not:

(*i*) $x^2 - 3\sqrt{3}\,x + 6 = 0;\ \sqrt{3}, -2\sqrt{3}$ (*ii*) $x^2 - \sqrt{2}\,x - 4 = 0;\ -\sqrt{2}, 2\sqrt{2}$

4. (*i*) If $-\frac{1}{2}$ is a solution of the equation $3x^2 + 2kx - 3 = 0$, find the value of k.

(*ii*) If $\frac{2}{3}$ is a solution of the equation $7x^2 + kx - 3 = 0$, find the value of k.

5. (*i*) If $\sqrt{2}$ is a root of the equation $kx^2 + \sqrt{2}\,x - 4 = 0$, find the value of k.

(*ii*) If a is a root of the equation $x^2 - (a + b)x + k = 0$, find the value of k.

6. If $\frac{2}{3}$ and -3 are the roots of the equation $px^2 + 7x + q = 0$, find the values of p and q.

5.2 SOLVING A QUADRATIC EQUATION BY FACTORISATION

Factorisation can be used to solve a quadratic equation. The equation $x^2 - 7x + 12 = 0$ can be written as $(x - 3)(x - 4) = 0$. This equation can be solved by using a property of real numbers called ***zero-product rule***.

❑ Zero-product rule

If a and b are two real numbers or algebraic expressions and if $ab = 0$, then either $a = 0$ or $b = 0$ or both $a = 0$ and $b = 0$.

Using the above rule, the solutions of the equation $(x - 3)(x - 4) = 0$ can be found by putting each factor equal to zero and then solving for x. Thus we get,

$$x - 3 = 0 \text{ or } x - 4 = 0$$

$$\Rightarrow \quad x = 3 \text{ or } x = 4.$$

Hence, the solutions of the equation $x^2 - 7x + 12 = 0$ are 3, 4.