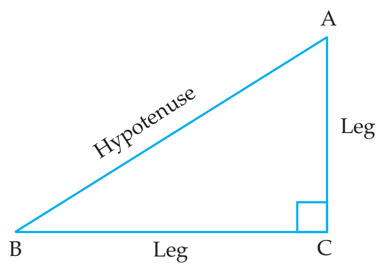
Pythagoras Theorem

11

INTRODUCTION

Pythagoras, a Greek philospher of sixth century B.C. discovered a very important and useful property of right angled triangles, named Pythagoras property. In a right angled triangle, the sides have special names. The side opposite to the right angle is called **hypotenuse** and the other two sides are called the **legs** of the right angled triangle. In the adjoining triangle ABC, ∠C = 90º. So, AB is its hypotenuse and BC and CA are the two legs. You are already familiar with the Pythagoras theorem from your earlier classes. In this chapter, we shall prove this theorem by using similarity of triangles. We shall also prove its converse and will learn their applications.

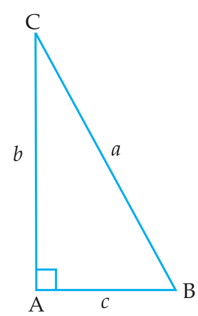


11.1 PYTHAGORAS THEOREM

In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Given. ABC is a right angled triangle at A *i.e.* ∠A = 90°, so that BC is its hypotenuse.

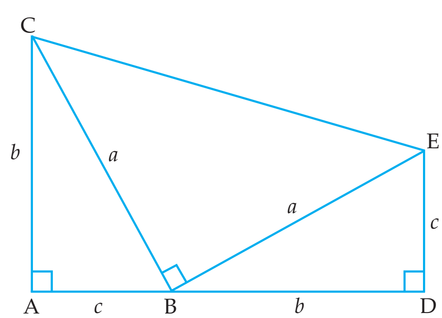
To prove. $BC^2 = CA^2 + AB^2$ *i.e.* $a^2 = b^2 + c^2$, where BC = a, CA = b and AB = c.



Construction. Extend the side AB to a point D such that BD = CA = b. At D, draw DE ⊥ AD and cut off DE = AB = c. Join CE.

Proof. In ΔABC and ΔDEB,

	CA	=	BD		(by construction)
	AB	=	DE		(by construction)
	∠A	=	∠D		(each = 90°)
∴	ΔABC	≅	ΔDEB		(SAS rule of congruency)
⇒	BC	=	BE		(c.p.c.t)
⇒	BE	=	a		(∵ BC = a)
and	∠ACB	=	∠DBE	…(i)	(c.p.c.t)



In ΔABC, $\angle A + \angle ABC + \angle ACB = 180°$

$\Rightarrow$ $90° + \angle ABC + \angle ACB = 180°$

$\Rightarrow$ $\angle ABC + \angle ACB = 90°$

$\Rightarrow$ $\angle ABC + \angle DEB = 90°$ ...(*ii*) (using (*i*))

Since sum of angles at a point on one side of a straight line is 180°,

$\therefore$ $\angle ABC + \angle CBE + \angle DBE = 180°$

$\Rightarrow$ $(\angle ABC + \angle DBE) + \angle CBE = 180°$

$\Rightarrow$ $90° + \angle CBE = 180°$ (using (*ii*))

$\Rightarrow$ $\angle CBE = 90°$

$\Rightarrow$ CBE is a right angled triangle at B.

Now, $\angle A + \angle D = 90° + 90° = 180°$

$\Rightarrow$ AC || DE ($\because$ sum of co-int. $\angle s = 180°$)

$\Rightarrow$ CADE is a trapezium.

From figure,

area of trapezium CADE = area of ΔCAB + area of ΔBDE + area of ΔCBE

$\Rightarrow$ $\frac{1}{2}(CA + ED) \times AD = \frac{1}{2}CA \times AB + \frac{1}{2}BD \times DE + \frac{1}{2}CB \times EB$

($\because$ area of a trapezium = $\frac{1}{2}$(sum of || sides × height and

area of a triangle = $\frac{1}{2}$ base × height)

$\Rightarrow$ $(b + c)(c + b) = bc + bc + a \times a$ ($\because AD = AB + BD = c + b$)

$\Rightarrow$ $(b + c)^2 = 2bc + a^2$

$\Rightarrow$ $b^2 + c^2 + 2bc = 2bc + a^2$

$\Rightarrow$ $b^2 + c^2 = a^2$

$\Rightarrow$ $CA^2 + AB^2 = BC^2$

Hence, $BC^2 = CA^2 + AB^2$

The above result is known as **Pythagoras Theorem**.

The above theorem was earlier given by an ancient mathematician **Baudhayan** (about 800 B.C.) in the following form:

The diagonal of a rectangle produces by itself the same area as produced by its both sides (i.e. length and breadth).

Therefore, the above theorem is sometimes also referred to as the ***Baudhayan Theorem***.

The converse of the Pythagoras theorem is also true. We record it as:

❒ Converse of Pythagoras Theorem

In a triangle, if square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to the first side is a right angle.

Given. In a triangle ABC, $BC^2 = AB^2 + AC^2$.

To prove. $\angle A = 90°$.

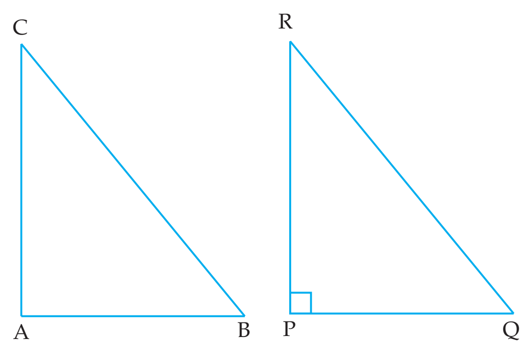
Construction. Construct a ΔPQR such that

$\angle P = 90°$, PQ = AB and PR = AC.

Proof. In ΔPQR, $\angle P = 90°$.

By Pythagoras theorem, we have

$QR^2 = PQ^2 + PR^2$



$\Rightarrow \quad QR^2 = AB^2 + AC^2$ (∵ PQ = AB and PR = AC)

But $BC^2 = AB^2 + AC^2$ (given)

$\therefore \quad QR^2 = BC^2 \Rightarrow QR = BC.$

In ΔABC and ΔPQR,

AB = PQ (by construction)

AC = PR (by construction)

and BC = QR (proved above)

$\therefore \quad \Delta ABC \cong \Delta PQR$ (SSS rule of congruency)

$\Rightarrow \quad \angle A = \angle P$ (c.p.c.t.)

$\Rightarrow \quad \angle A = 90°$ ($\because \angle P = 90°$, by construction)

Illustrative Examples

Example 1. *Lengths of sides of triangles are given below. Determine which of them are right triangles. In case of a right triangle, write the length of its hypotenuse:*

(i) 7 cm, 24 cm, 25 cm *(ii) 50 cm, 80 cm, 100 cm*

Solution. Choose the greatest length. Check whether the square of greatest length is equal to the sum of squares of other two lengths.

(*i*) Here, greatest length is 25 cm and other lengths are 7 cm, 24 cm.

Note that $25^2 = 625$ and $7^2 + 24^2 = 49 + 576 = 625$.

Thus, $25^2 = 7^2 + 24^2$.

Therefore, the triangle with given lengths of sides is a right triangle and the length of its hypotenuse is 25 cm.

(*ii*) Here, greatest length is 100 cm and other lengths are 50 cm, 80 cm.

Note that $50^2 + 80^2 = 2500 + 6400 = 8900 \neq 100^2$.

Therefore, the triangle with given lengths of sides is not a right triangle.

Example 2. *ABC is an isosceles triangle with AC = BC. If $AB^2 = 2AC^2$, prove that ABC is a right triangle.*

Solution. Given $AB^2 = 2AC^2 \Rightarrow AB^2 = AC^2 + AC^2$

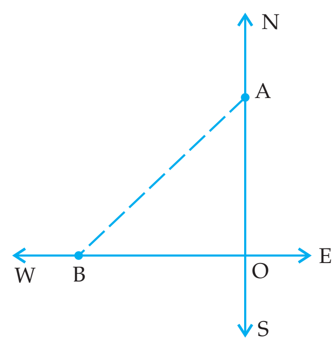
$\Rightarrow \quad AB^2 = AC^2 + BC^2$ (∵ AC = BC, given)

$\Rightarrow \quad \angle C = 90°$ (converse of Pythagoras theorem)

Hence, ΔABC is a right triangle.

Example 3. *An aeroplane leaves an airport and flies due north at a speed of 1000 km per hour. At the same time, another aeroplane leaves the same airport and flies due west at a speed of 1200 km per hour. How far apart will be the two planes after $1\frac{1}{2}$ hours?*

Solution. Two aeroplanes leave an airport O at the same time. Let A and B be the positions of the aeroplanes after $1\frac{1}{2}$ hours *i.e.* $\frac{3}{2}$ hours.



OA = distance travelled in $\frac{3}{2}$ hours by the aeroplane due north

$= \left(1000 \times \frac{3}{2}\right)$ km = 1500 km, OB = distance travelled in $\frac{3}{2}$ hours by the aeroplane due west

$= \left(1200 \times \frac{3}{2}\right)$ km = 1800 km

In ΔAOB, $\angle O = 90°$. By Pythagoras theorem, we get

$AB^2 = OA^2 + OB^2 = 1500^2 + 1800^2 = (300)^2 (5^2 + 6^2)$

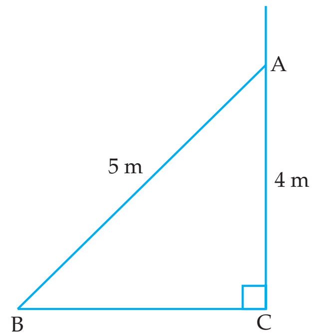
$\Rightarrow$ $AB = 300\sqrt{61}$ km

Hence, the distance between two aeroplanes = $300\sqrt{61}$ km.

Example 4. *A 5 m long ladder is placed leaning towards a vertical wall such that it reaches the wall at a point 4 m high. If the foot of the ladder is moved 1·6 m towards the wall, then find the distance by which the top of the ladder would slide upwards on the wall.*

Solution. The point C is the base of wall.

Originally, let the top of ladder reach the wall at the point A.



In ΔABC, $\angle C = 90°$. By Pythagoras theorem, we get

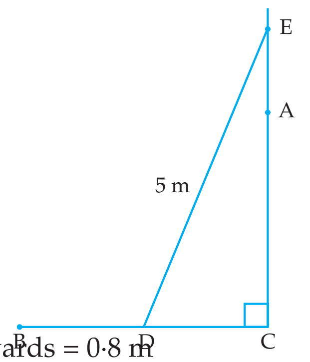
$\therefore$ $BC^2 + AC^2 = AB^2$

$\Rightarrow$ $BC^2 + 4^2 = 5^2$

$\Rightarrow$ $BC^2 = 25 - 16 = 9$

$\Rightarrow$ $BC = 3$ m.

Now, the foot of ladder is moved 1·6 m towards the wall. Let D be new position of foot of ladder and E be the new position of its top.



DC = BC – BD = 3 m – 1·6 m = 1·4 m

In ΔECD, $\angle C = 90°$. By Pythagoras theorem, we get

$\therefore$ $EC^2 + DC^2 = DE^2$

$\Rightarrow$ $EC^2 + (1{\cdot}4)^2 = 5^2$

$\Rightarrow$ $EC^2 = 25 - 1{\cdot}96 = 23{\cdot}04$

$\Rightarrow$ $EC = 4{\cdot}8$ m.

$\therefore$ AE = EC – AC = 4·8 m – 4 m = 0·8 m

Hence, the distance by which the top of ladder would slide upwards = 0·8 m

Example 5. *In ΔABC, $\angle B = 90°$ and D is mid-point of BC. Prove that $AC^2 = AD^2 + 3CD^2$.*

Solution. As D is mid-point of BC,

BD = CD and BC = 2CD.

In ΔABD, $\angle ABD = 90°$,

$\therefore$ $AD^2 = AB^2 + BD^2$ (Pythagoras theorem)

$\Rightarrow$ $AB^2 = AD^2 - BD^2$ …(*i*)

In ΔABC, $\angle B = 90°$,

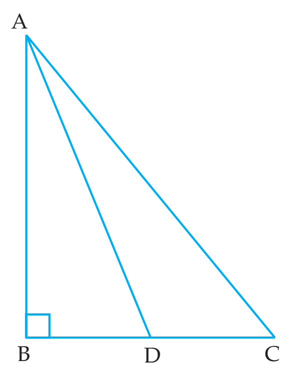
$\therefore$ $AC^2 = AB^2 + BC^2$

$\Rightarrow$ $AC^2 = (AD^2 - BD^2) + BC^2$ (using (*i*))

$\Rightarrow$ $AC^2 = AD^2 - CD^2 + (2CD)^2$ ($\because$ BD = CD, BC = 2CD)

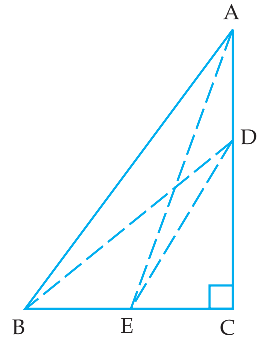
$\Rightarrow$ $AC^2 = AD^2 - CD^2 + 4CD^2$

$\Rightarrow$ $AC^2 = AD^2 + 3CD^2$.



Example 6. *D and E are points on the sides CA and CB respectively of a triangle ABC right angled at C. Prove that* $AE^2 + BD^2 = AB^2 + DE^2$.

Solution. In ΔABC, $\angle C = 90°$



$\therefore \quad AB^2 = AC^2 + BC^2 \qquad \ldots(i)$

In ΔECD, $\angle ECD = 90°$

$\therefore \quad DE^2 = CD^2 + EC^2 \qquad \ldots(ii)$

In ΔAEC, $\angle ACE = 90°$

$\therefore \quad AE^2 = AC^2 + EC^2 \qquad \ldots(iii)$

In ΔBCD, $\angle BCD = 90°$

$\therefore \quad BD^2 = BC^2 + CD^2 \qquad \ldots(iv)$

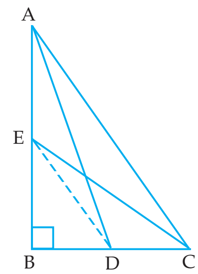
Adding (*iii*) and (*iv*), we get

$$AE^2 + BD^2 = (AC^2 + BC^2) + (CD^2 + EC^2)$$

$$= AB^2 + DE^2 \qquad \text{(using } (i) \text{ and } (ii))$$

Example 7. *ABC is a right angled triangle at B. If D and E are mid-points of sides BC and AB respectively, prove that* $AD^2 + CE^2 = 5\,DE^2$.

Solution. As D is mid-point of BC, BC = 2BD.



Also, as E is mid-point of AB, AB = 2BE.

In ΔABD, $\angle B = 90°$, by Pythagoras theorem,

$$AD^2 = AB^2 + BD^2$$

$$= (2BE)^2 + BD^2 \qquad (\because AB = 2BE)$$

$$\Rightarrow \quad AD^2 = 4BE^2 + BD^2 \qquad \ldots(i)$$

In ΔEBC, $\angle B = 90°$, by Pythagoras theorem,

$$CE^2 = BE^2 + BC^2$$

$$= BE^2 + (2BD)^2 \qquad (\because BC = 2BD)$$

$$\Rightarrow \quad CE^2 = BE^2 + 4BD^2 \qquad \ldots(ii)$$

Adding (*i*) and (*ii*), we get

$$AD^2 + CE^2 = 5(BE^2 + BD^2) \qquad \ldots(iii)$$

In ΔEBD, $\angle B = 90°$, by Pythagoras theorem,

$$DE^2 = BE^2 + BD^2 \qquad \ldots(iv)$$

From (*iii*) and (*iv*), we get

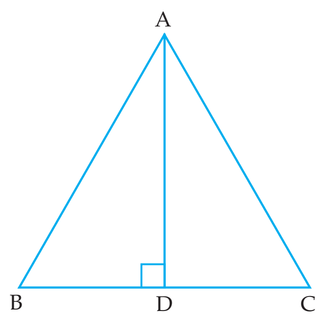
$AD^2 + CE^2 = 5DE^2$, as required.

Example 8. *ABC is an equilateral triangle of side 2a. Find each of its altitude.*

Solution. Given ABC is an equilateral triangle of side $2a$.

Draw $AD \perp BC$

In ΔABD and ΔACD,



$$AB = AC \qquad \text{(given)}$$

$$\angle ADB = \angle ADC \qquad (\text{each} = 90°, AD \perp BC)$$

$$AD = AD \qquad \text{(common)}$$

$$\therefore \quad \Delta ABD \cong \Delta ACD \qquad \text{(RHS rule of congruency)}$$

$$\Rightarrow \quad BD = DC \qquad \text{(c.p.c.t.)}$$

$$\Rightarrow \quad BD = \frac{1}{2}BC = \frac{1}{2}.2a = a.$$

In ΔABD, $\angle ADB = 90°$,

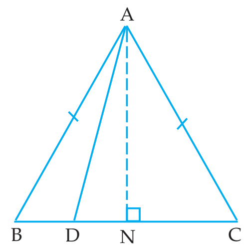
$$\therefore \quad AB^2 = AD^2 + BD^2 \qquad \text{(Pythagoras theorem)}$$

$\Rightarrow \quad (2a)^2 = AD^2 + a^2 \Rightarrow AD^2 = 3a^2 \;\Rightarrow AD = \sqrt{3}\,a.$

In an equilateral triangle, all altitudes are equal.

Hence, the length of each altitude = $\sqrt{3}\,a$.

Example 9. *ABC is a triangle in which AB = AC and D is a point on BC. Prove that* $AB^2 - AD^2 = BD \times DC$.



Solution. Draw AN ⊥ BC.

In ΔABN and ΔANC,

$AB = AC$ (given)

$\angle ANB = 90° = \angle ANC$

and AN is common,

$\therefore \quad \Delta ABN \cong \Delta ANC$ (RHS rule of congruency)

$\Rightarrow \quad BN = NC.$

In ΔABN, $\angle N = 90°$,

$\therefore \quad AB^2 = AN^2 + BN^2$ ...(*i*)

In ΔADN, $\angle N = 90°$,

$\therefore \quad AD^2 = AN^2 + DN^2$ ...(*ii*)

Subtracting (*ii*) from (*i*), we get

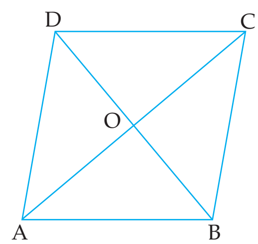
$$AB^2 - AD^2 = BN^2 - DN^2$$

$$= (BN + DN)\,(BN - DN) \qquad (\because\ BN = NC)$$

$$= (NC + DN) \times BD = DC \times BD$$

Hence, $AB^2 - AD^2 = BD \times DC$.

Example 10. *Prove that the sum of the squares on the sides of a rhombus is equal to the sum of squares on its diagonals.*



Solution. Let ABCD be a rhombus whose diagonals AC and BD intersect at the point O.

As the diagonals of a rhombus bisect each other at right angles, $\angle AOB = 90°$ and $OA = \frac{1}{2}AC$, $OB = \frac{1}{2}BD$.

In ΔOAB, $\angle AOB = 90°$,

$\therefore \quad AB^2 = OA^2 + OB^2$ (Pythagoras theorem)

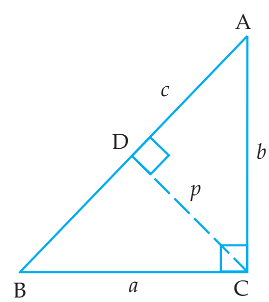
$\Rightarrow \quad AB^2 = \left(\frac{1}{2}AC\right)^2 + \left(\frac{1}{2}BD\right)^2 \Rightarrow 4AB^2 = AC^2 + BD^2$

But $AB = BC = CD = DA$ ($\because$ in a rhombus, sides are equal)

$\therefore \quad AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2.$

Example 11. *ABC is a right triangle, right angled at C. If p is the length of perpendicular from C to AB and a, b, c have usual meanings, then prove that*

(*i*) $pc = ab$ (*ii*) $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.



Solution. (*i*) Area of ΔABC = $\frac{1}{2}AB \times CD = \frac{1}{2}BC \times AC$

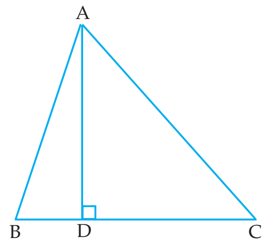
$\Rightarrow \quad c \times p = ab.$

(*ii*) In ΔABC, $\angle C = 90°$, so $c^2 = a^2 + b^2$

$\Rightarrow \quad \left(\frac{ab}{p}\right)^2 = a^2 + b^2$ (using part (*i*))

$\Rightarrow \quad \frac{1}{p^2} = \frac{a^2 + b^2}{a^2 b^2} = \frac{1}{b^2} + \frac{1}{a^2}.$

Example 12. *In the adjoining figure, AD ⊥ BC. If D divides BC in the ratio 1 : 3, prove that $2AC^2 = 2AB^2 + BC^2$.*



Solution. Given D divides BC in the ratio 1 : 3,

$\therefore \quad \frac{BD}{DC} = \frac{1}{3} \Rightarrow DC = 3BD$

$\therefore \quad BC = BD + DC = BD + 3BD = 4BD$

$\Rightarrow \quad BD = \frac{1}{4} BC.$

In ΔADC, ∠D = 90°,

$\therefore \quad AC^2 = AD^2 + DC^2$

$= AD^2 + (3BD)^2 = AD^2 + 9BD^2 \quad \ldots(i)$

In ΔABD, ∠D = 90°,

$\therefore \quad AB^2 = AD^2 + BD^2 \Rightarrow AD^2 = AB^2 - BD^2 \quad \ldots(ii)$

From (*i*) and (*ii*), we get

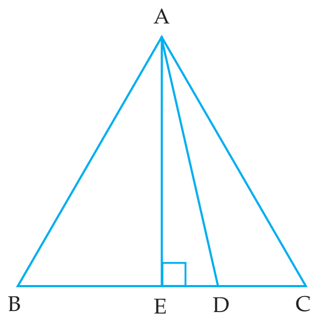
$AC^2 = (AB^2 - BD^2) + 9BD^2$

$= AB^2 + 8BD^2 = AB^2 + 8 . \left(\frac{1}{4}BC\right)^2$

$= AB^2 + \frac{1}{2}BC^2$

$\Rightarrow \quad 2AC^2 = 2AB^2 + BC^2.$

Example 13. *In an equilateral triangle ABC, a point D is taken on base BC such that BD : DC = 2 : 1. Prove that $9AD^2 = 7AB^2$.*



Solution. As ΔABC is equilateral, BC = AB.

Given BD : DC = 2 : 1 $\Rightarrow BD = \frac{2}{3}BC = \frac{2}{3}AB.$

Draw AE ⊥ BC, then E is mid-point of BC,

so $BE = \frac{1}{2}BC = \frac{1}{2}AB.$

From fig., $ED = BD - BE = \frac{2}{3}AB - \frac{1}{2}AB = \frac{1}{6}AB.$

In ΔABE, ∠AEB = 90°,

$\therefore \quad AB^2 = AE^2 + BE^2 \quad \ldots(i)$

In ΔAED, ∠AED = 90°,

$\therefore \quad AD^2 = AE^2 + ED^2 \quad \ldots(ii)$

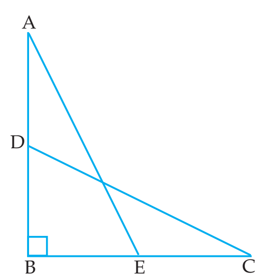
Subtracting (*ii*) from (*i*), we get

$$AB^2 - AD^2 = BE^2 - ED^2 = \left(\frac{1}{2}AB\right)^2 - \left(\frac{1}{6}AB\right)^2 = \left(\frac{1}{4} - \frac{1}{36}\right)AB^2 = \frac{2}{9}AB^2$$

$\Rightarrow \quad AD^2 = AB^2 - \frac{2}{9}AB^2 = \frac{7}{9}AB^2$

$\Rightarrow \quad 9AD^2 = 7AB^2.$

Example 14. *In the adjoining figure, AE = DC = 13 cm, BE = 5 cm, ∠ABC = 90° and AD = EC = x cm. Calculate the length of AB and the value of x.*



Solution. In ΔABE, ∠B = 90°,

$\therefore \quad AE^2 = AB^2 + BE^2$

$\Rightarrow \quad AB^2 = AE^2 - BE^2$

$\Rightarrow \quad AB^2 = (13)^2 - (5)^2 \quad (\because AE = 13 \text{ cm}, BE = 5 \text{ cm})$

$\Rightarrow \quad AB^2 = 169 - 25 = 144$

$\Rightarrow \quad AB = 12$ cm.

From figure, BD = AB – AD = $(12 - x)$ cm (∵ AB = 12 cm, AD = x cm)

and BC = BE + EC = $(5 + x)$ cm.

In ΔBCD, ∠B = 90°,

$\therefore \quad CD^2 = BD^2 + BC^2$

$\Rightarrow \quad (13)^2 = (12 - x)^2 + (5 + x)^2$

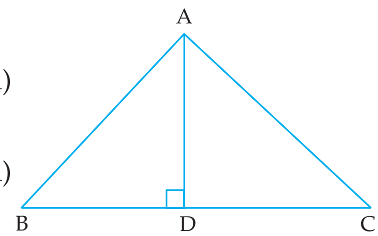
$\Rightarrow \quad 169 = 144 + x^2 - 24x + 25 + x^2 + 10x$

$\Rightarrow \quad 169 = 169 + 2x^2 - 14x$

$\Rightarrow \quad 2x^2 - 14x = 0 \Rightarrow 2x(x - 7) = 0$

$\Rightarrow \quad x = 7$ cm. $(\because x \neq 0)$

Example 15. *In ΔABC, AD ⊥ BC such that $AD^2 = BD \times DC$. Using Pythagoras theorem and its converse, prove that ΔABC is right angled at A.*



Solution. In ΔABD, ∠ADB = 90°,

$\therefore \quad AB^2 = AD^2 + BD^2$ ...(*i*) (Pythagoras theorem)

In ΔACD, ∠ADC = 90°,

$\therefore \quad AC^2 = AD^2 + DC^2$ ...(*ii*) (Pythagoras theorem)

On adding (*i*) and (*ii*), we get

$AB^2 + AC^2 = 2AD^2 + BD^2 + CD^2$

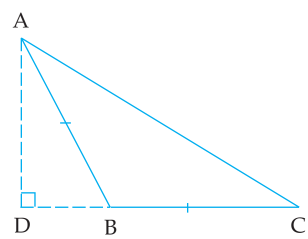
$= 2\,BD \times DC + DB^2 + DC^2$ (∵ $AD^2 = BD \times DC$, given)

$= (BD + DC)^2 = BC^2$

∴ By converse of Pythagoras theorem, ΔABC is right angled at A.

Example 16. *In the adjoining figure, AB = BC and AD ⊥ CB (produced). Prove that*

$AC^2 = 2BC \times CD.$



Solution. In ΔADC, ∠ADC = 90°, so $AC^2 = AD^2 + DC^2$ ...(*i*)

In ΔADB, ∠ADB = 90°, so $AB^2 = AD^2 + DB^2$ ...(*ii*)

Subtracting (*ii*) from (*i*), we get

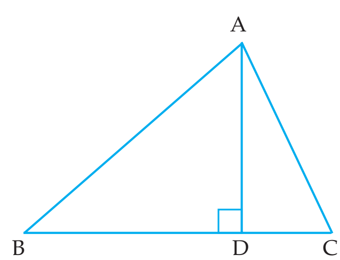
$AC^2 - AB^2 = DC^2 - DB^2$

$\Rightarrow \quad AC^2 - BC^2 = (DB + BC)^2 - DB^2$ (∵ AB = BC)

$\Rightarrow \quad AC^2 = BC^2 + BC^2 + 2DB \times BC = 2BC^2 + 2DB \times BC$

$= 2BC(BC + DB) = 2BC \times DC.$

Example 17. *In the adjoining figure, ∠B of ΔABC is an acute angle and AD ⊥ BC. Prove that $AC^2 = AB^2 + BC^2 - 2BC \times BD$.*



Solution. In ΔABD, ∠ADB = 90°,

$\therefore \quad AB^2 = AD^2 + BD^2$ ...(*i*)

In ΔADC, ∠ADC = 90°,

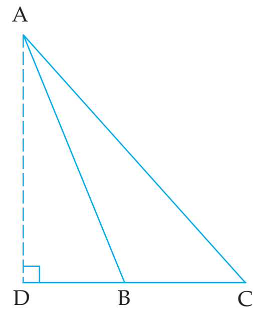
$\therefore \quad AC^2 = AD^2 + DC^2$

$\Rightarrow \quad AC^2 = AD^2 + (BC - BD)^2 = AD^2 + BC^2 + BD^2 - 2BC \times BD$

$= (AD^2 + BD^2) + BC^2 - 2BC \times BD$

$= AB^2 + BC^2 - 2BC \times BD$ (using (*i*))

Example 18. *In the adjoining figure, $\angle B$ of ΔABC is obtuse and $AD \perp BC$ (produced). Prove that $AC^2 = AB^2 + BC^2 + 2BC \times BD$.*



Solution. In ΔADB, $\angle ADB = 90°$,

$\therefore \quad AB^2 = AD^2 + DB^2 \qquad \ldots(i)$

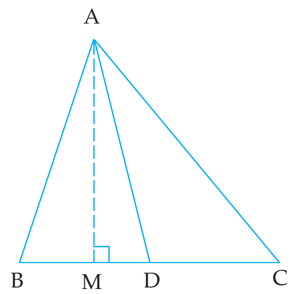
In ΔADC, $\angle ADC = 90°$,

$\therefore \quad AC^2 = AD^2 + DC^2$

$\Rightarrow \quad AC^2 = AD^2 + (DB + BC)^2 = AD^2 + DB^2 + BC^2 + 2BC \times BD$

$\Rightarrow \quad AC^2 = AB^2 + BC^2 + 2BC \times BD \qquad$ (using (*i*))

Example 19. *In the adjoining figure, AD is median of ΔABC and $AM \perp BC$. Prove that*



(*i*) $AC^2 = AD^2 + BC \times DM + \frac{1}{4}BC^2$

(*ii*) $AB^2 = AD^2 - BC \times DM + \frac{1}{4}BC^2$

(*iii*) $AB^2 + AC^2 = 2AD^2 + \frac{1}{2}BC^2$

(*iv*) $AB^2 + AC^2 = 2(AD^2 + BD^2)$.

Solution. Note that in the given figure, AC > AB.

As AD is median, BD = DC.

$\Rightarrow \quad BD = DC = \frac{1}{2}BC.$

In ΔAMD, $\angle AMD = 90°$,

$\therefore \quad AD^2 = AM^2 + MD^2 \Rightarrow AM^2 = AD^2 - MD^2 \qquad \ldots(1)$

(*i*) In ΔAMC, $\angle AMC = 90°$,

$\therefore \quad AC^2 = AM^2 + MC^2$

$= (AD^2 - MD^2) + (MD + DC)^2 \qquad$ (using (1))

$= AD^2 - MD^2 + \left(MD + \frac{1}{2}BC\right)^2 \qquad \left(\because DC = \frac{1}{2}BC\right)$

$\Rightarrow \quad AC^2 = AD^2 + MD \times BC + \frac{1}{4}BC^2 \qquad \ldots(2)$

(*ii*) In ΔABM, $\angle AMB = 90°$,

$\therefore \quad AB^2 = AM^2 + BM^2$

$= (AD^2 - MD^2) + (BD - MD)^2 \qquad$ (using (1))

$= AD^2 - MD^2 + \left(\frac{1}{2}BC - MD\right)^2 \qquad \left(\because BD = \frac{1}{2}BC\right)$

$\Rightarrow \quad AB^2 = AD^2 - MD \times BC + \frac{1}{4}BC^2 \qquad \ldots(3)$

(*iii*) On adding (2) and (3), we get

$$AB^2 + AC^2 = 2AD^2 + \frac{1}{2}BC^2$$

(*iv*) From part (*iii*), we have

$$AB^2 + AC^2 = 2AD^2 + \frac{1}{2}(2BD)^2 \qquad \left(\because BD = \frac{1}{2}BC\right)$$

$\Rightarrow \quad AB^2 + AC^2 = 2AD^2 + \frac{1}{2}.4\,BD^2$

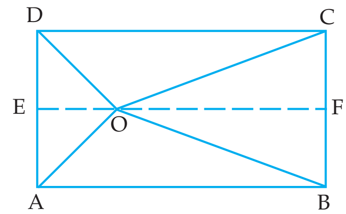
$\Rightarrow \quad AB^2 + AC^2 = 2(AD^2 + BD^2).$

Example 20. *If O is any point in the interior of a rectangle ABCD. Prove that*

$OA^2 + OC^2 = OB^2 + OD^2$.

Hence, find the length of OD, if the lengths of OA, OB and OC are 3 cm, 4 cm and 5 cm respectively.

Solution. Through O, draw EF || AB.



As ABCD is a rectangle, AD ⊥ AB.

Since EF || AB and AD ⊥ AB, therefore, EF ⊥ AD.

Similarly, EF ⊥ BC.

In ΔOEA, ∠OEA = 90°,

$OA^2 = AE^2 + OE^2$ …(*i*)

In ΔOFC, ∠OFC = 90°,

$OC^2 = FC^2 + OF^2$ …(*ii*)

Adding (*i*) and (*ii*), we get

$OA^2 + OC^2 = OE^2 + OF^2 + AE^2 + FC^2$ …(*iii*)

In ΔOBF, ∠OFB = 90°, $OB^2 = OF^2 + BF^2$ …(*iv*)

In ΔEOD, ∠DEO = 90°, $OD^2 = OE^2 + ED^2$ …(*v*)

Adding (*iv*) and (*v*), we get

$OB^2 + OD^2 = OF^2 + OE^2 + BF^2 + ED^2$ …(*vi*)

But AE = BF and FC = ED.

∴ From (*iii*) and (*vi*), we get

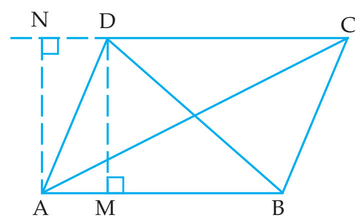
$OA^2 + OC^2 = OB^2 + OD^2$.

Further, OA = 3 cm, OB = 4 cm and OC = 5 cm

∴ $3^2 + 5^2 = 4^2 + OD^2 \Rightarrow OD^2 = 18 \Rightarrow OD = 3\sqrt{2}$ cm.

Example 21. *Prove that the sum of the squares of the diagonals of a parallelogram is equal to the sum of squares of its sides.*

Solution. Let ABCD be a parallelogram, then AC and BD are its diagonals.



So, we are required to prove that

$AC^2 + BD^2 = AB^2 + BC^2 + CD^2 + AD^2$.

Draw DM ⊥ AB and AN ⊥ DC (produced), AMDN is a rectangle

⇒ AM = ND …(*i*)

In ΔAND, ∠AND = 90°, so $AD^2 = AN^2 + ND^2$ …(*ii*)

In ΔANC, ∠ANC = 90°,

∴ $AC^2 = AN^2 + NC^2 = AN^2 + (ND + CD)^2$

$= AN^2 + ND^2 + CD^2 + 2ND \times CD$

⇒ $AC^2 = AD^2 + CD^2 + 2\,ND \times CD$ …(*iii*) (using (*ii*))

In ΔAMD, ∠AMD = 90°, so $AD^2 = AM^2 + MD^2$ …(*iv*)

In ΔDMB, ∠DMB = 90°,

∴ $BD^2 = MD^2 + MB^2 = MD^2 + (AB - AM)^2$

$= MD^2 + AM^2 + AB^2 - 2AM \times AB$

$= AD^2 + AB^2 - 2AM \times AB$ (using (*iv*))

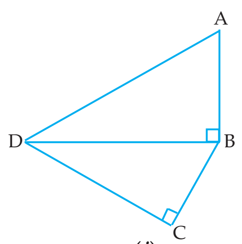
$= AD^2 + AB^2 - 2ND \times CD$ …(*v*) (∵ AM = ND and AB = CD)

On adding (*iii*) and (*v*), we get

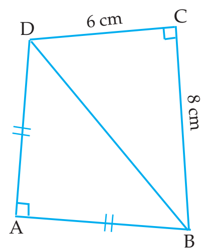
$AC^2 + BD^2 = 2AD^2 + CD^2 + AB^2 = AD^2 + BC^2 + AB^2 + CD^2$ (∵ AD = BC)

Exercise 11

1. Lengths of sides of triangles are given below. Determine which of them are right triangles. In case of a right triangle, write the length of its hypotenuse:
   (*i*) 3 cm, 8 cm, 6 cm (*ii*) 13 cm, 12 cm, 5 cm
   (*iii*) 1·4 cm, 4·8 cm, 5 cm
2. Foot of a 10 m long ladder leaning against a vertical well is 6 m away from the base of the wall. Find the height of the point on the wall where the top of the ladder reaches.
3. A guy attached a wire 24 m long to a vertical pole of height 18 m and has a stake attached to the other end. How far from the base of the pole should the stake be driven so that the wire will be taught?
4. Two poles of heights 6 m and 11 m stand on a plane ground. If the distance between their feet is 12 m, find the distance between their tops.
5. In a right-angled triangle, if hypotenuse is 20 cm and the ratio of the other two sides is 4 : 3, find the sides.
6. If the sides of a triangle are in the ratio 3 : 4 : 5, prove that it is right-angled triangle.
7. For going to a city B from city A, there is route via city C such that AC ⊥ CB, AC = $2x$ km and CB = $2(x + 7)$ km. It is proposed to construct a 26 km highway which directly connects the two cities A and B. Find how much distance will be saved in reaching city B from city A after the construction of highway.
8. The hypotenuse of a right triangle is 6 m more than twice the shortest side. If the third side is 2 m less than the hypotenuse, find the sides of the triangle.
9. ABC is an isosceles triangle right angled at C. Prove that $AB^2 = 2AC^2$.
10. In a triangle ABC, AD is perpendicular to BC. Prove that $AB^2 + CD^2 = AC^2 + BD^2$.
11. In ΔPQR, PD ⊥ QR, such that D lies on QR. If PQ = a, PR = b, QD = c and DR = d, prove that $(a + b)(a - b) = (c + d)(c - d)$.
12. ABC is an isosceles triangle with AB = AC = 12 cm and BC = 8 cm. Find the altitude on BC and hence calculate its area.
13. Find the area and the perimeter of a square whose diagonal is 10 cm long.
14. (*a*) In figure (*i*) given below, ABCD is a quadrilateral in which AD = 13 cm, DC = 12 cm, BC = 3 cm, ∠ABD = ∠BCD = 90°. Calculate the length of AB.
    (*b*) In figure (*ii*) given below, ABCD is a quadrilateral in which AB = AD, ∠A = 90° = ∠C, BC = 8 cm and CD = 6 cm. Find AB and calculate the area of ΔABD.



(*i*)



(*ii*)

15. (*a*) In figure (*i*) given below, AB = 12 cm, AC = 13 cm, CE = 10 cm and DE = 6 cm. Calculate the length of BD.
    (*b*) In figure (*ii*) given below, ∠PSR = 90°, PQ = 10 cm, QS = 6 cm and RQ = 9 cm. Calculate the length of PR.