GEOMETRY

Triangles

9

INTRODUCTION

In previous classes, we have studied about triangles and various types of triangles on the basis of sides and on the basis of angles. We have also studied the following:

(*i*) angles sum property of a triangle and exterior angle property

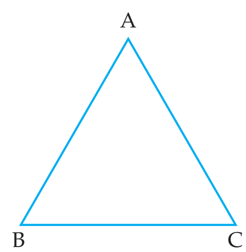
(*ii*) congruence of triangles

(*iii*) inequalities in a triangle.

In this chapter, we shall review these and shall study about congruence of triangles in detail, rules of congruency, some more properties of triangles and inequalities in a triangle.

9.1 TRIANGLE

A triangle is a closed curve formed by three line segments. 'Tri' means 'three'. A triangle has three sides, three angles and three vertices.



The adjoining figure shows a triangle ABC. The line segments AB, BC and CA are called its **sides**. The angles ∠A, ∠B and ∠C are called its **interior angles** or simply **angles**. The points A, B and C are called its **vertices**. Three sides and the three angles are called its six **elements**.

In the above figure, look at the vertex A. It is the point of intersection of the sides AB and AC, BC is the remaining side. We say that vertex A and side BC are opposite to each other. Also ∠A and side BC are opposite to each other.

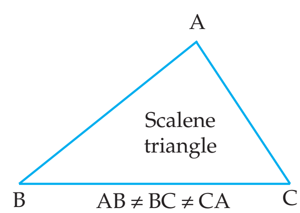
Similarly, vertex B and side CA are opposite to each other; ∠B and side CA are opposite to each other. Same can be said about vertex C, ∠C and side AB.

9.1.1 Types of triangles

Types of triangles on the basis of sides.

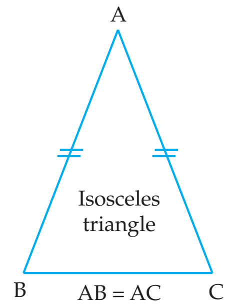
(*i*) **Scalene triangle.** *If all the sides of a triangle are unequal, it is called a* ***scalene triangle****.*

In the adjoining diagram, AB ≠ BC ≠ CA, so ΔABC is a scalene triangle.



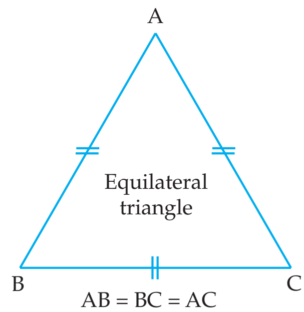
(*ii*) **Isosceles triangle.** *If any two sides of a triangle are equal, it is called an* ***isosceles triangle****.*

In the adjoining diagram, AB = AC, so ΔABC is an isosceles triangle. Usually, equal sides are indicated by putting marks on each of them.



(*iii*) **Equilateral triangle.** *If all the three sides of a triangle are equal, it is called an* ***equilateral triangle****.*

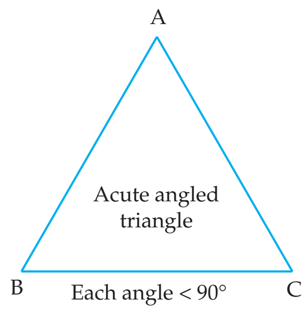
In the adjoining diagram, AB = BC = AC, so ΔABC is an equilateral triangle.



Types of triangles on the basis of angles.

(*i*) **Acute angled triangle.** *If all the three angles of a triangle are acute (less than 90°), it is called an* ***acute-angled triangle.***

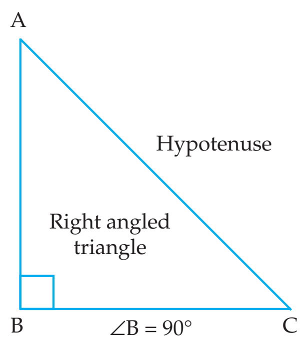
In the adjoining diagram, each angle is less than 90°, so ΔABC is an acute angled triangle.



(*ii*) **Right-angled triangle.** *If one angle of a triangle is a right angle (= 90°), it is called a* ***right-angled triangle****.*

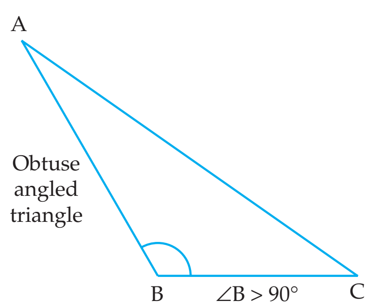
In a right angled triangle, the side opposite to right angle is called **hypotenuse.**

In the adjoining diagram, ∠B = 90°, so ΔABC is a right angled triangle and side AC is the hypotenuse.



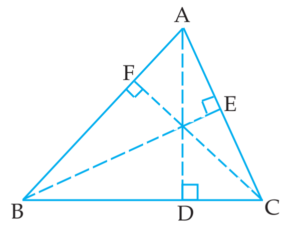
(*iii*) **Obtuse angled triangle.** *If one angle of a triangle is obtuse (greater than 90°), it is called an* ***obtuse-angled triangle****.*

In the adjoining diagram, ∠B is obtuse (greater than 90°), so ΔABC is an obtuse angled triangle.



9.1.2 Some terms connected with a triangle

Orthocentre. *Perpendicular from a vertex of a triangle to the opposite side is called an* ***altitude*** *of the triangle.*

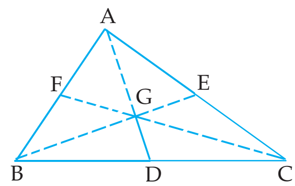


In the adjoining figure, AD ⊥ BC, so AD is an altitude of ΔABC.

A triangle has three altitudes.

In fact, all the three altitudes of a triangle pass through the same point and the point of concurrence is called the ***orthocentre*** of the triangle.

Centroid. The straight line joining a vertex of a triangle to the mid-point of the opposite side is called a ***median*** of the triangle.



In the adjoining figure, D is mid-point of BC, so AD is a median of ΔABC.

A triangle has three medians.

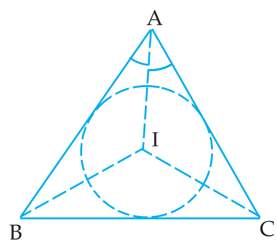
In fact, all the three medians of a triangle pass through the same point and the point of concurrence is called the ***centroid*** of the triangle.

The centroid of a triangle divides every median in the ratio of 2 : 1. Thus, if G is the centroid of ΔABC, then

AG : GD = 2 : 1, BG : GE = 2 : 1 and CG : GF = 2 : 1.

Incentre and incircle

Line bisecting an (interior) angle of a triangle is called the (internal) ***bisector*** of the angle of the triangle.



In the adjoining figure, ∠BAI = ∠IAC, so AI is the (internal) bisector of ∠A.

A triangle has three internal bisectors of its angles.

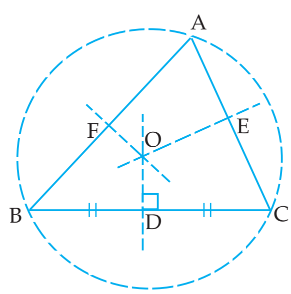
In fact, all the three (internal) bisectors of the angles of a triangle pass through the same point and the point of concurrence is called the ***incentre*** of the triangle.

In the above figure, IA, IB and IC are the (internal) bisectors of ∠A, ∠B and ∠C respectively. So I is the incentre of ΔABC.

Moreover, incentre is the centre of a circle which touches all the sides of ΔABC and this circle is called ***incircle*** of ΔABC.

Circumcentre and circumcircle

Line bisecting a side of a triangle and perpendicular to it is called the ***right bisector*** of the side of the triangle.



In the adjoining figure, D is mid-point of BC and OD ⊥ BC, so OD is the right bisector of the side BC.

A triangle has three right bisectors of its sides.

In fact, all the three right bisectors of the sides of a triangle pass through the same point and the point of concurrence is called the ***circumcentre*** of the triangle.

In the above diagram, OD, OE and OF are the right bisectors of the sides BC, CA and AB respectively of ΔABC. So O is the circumcentre of ΔABC.

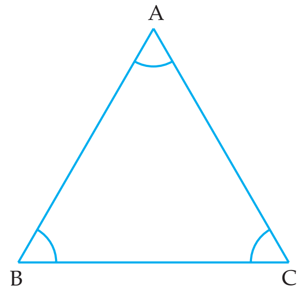
Moreover, circumcentre is the centre of a circle which passes through the vertices of ΔABC and this circle is called ***circumcircle*** of ΔABC.

9.1.3 Angles sum property of a triangle

The sum of angles of a triangle is 180°.

In the adjoining figure, ABC is a triangle.

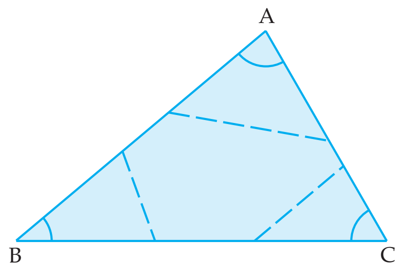
∠A + ∠B + ∠C = 180°.



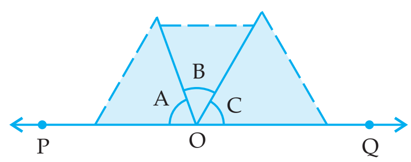
Activity

To verify that sum of angles of a triangle is 180° by cutting and pasting.

Steps



1. Take a sheet of paper and draw any triangle ABC on it and cut off the three angles *i.e.* ∠A, ∠B and ∠C along the dotted lines (as shown in the adjoining figure).



2. Draw any line PQ on the sheet of paper and mark a point O on it. Paste the cut outs of ∠A, ∠B and ∠C on the line PQ such that their vertices A, B and C all fall at the point O (as shown in the adjoining figure).

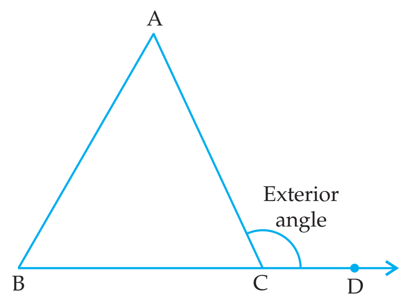
   Note that the outer arms of ∠A and ∠C coincide with the line PQ. The three angles now constitute one angle *i.e.* ∠POQ.

   But POQ is a straight angle,

   ∴ ∠A + ∠B + ∠C = 180°.

9.1.4 An exterior angle property of a triangle

Let ABC be a triangle and its side BC be produced to D, then ∠ACD is called an ***exterior angle*** at C. The two interior angles of the triangle that are opposite to the exterior ∠ACD are called its ***interior opposite angles*** or ***remote interior angles***. Thus, ∠ABC and ∠BAC of ΔABC are interior opposite angles of the exterior ∠ACD.



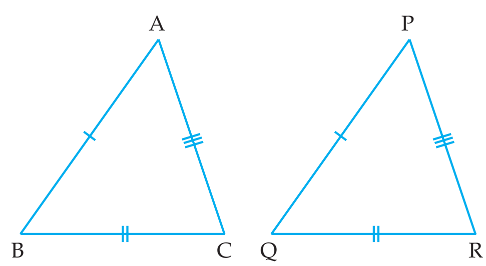
An exterior angle of a triangle is equal to sum of its interior opposite angles.

In the above figure, ∠ACD = ∠A + ∠B.

9.2 CONGRUENCE OF TRIANGLES

Two triangles are called ***congruent*** *if and only if they have exactly the same shape and the same size.*

In the adjoining figure, two triangles ABC and PQR are congruent. It means that the sketch of one triangle can be slided onto the sketch of the other so that they fit each other exactly *i.e.* when one triangle is superimposed on the other, they cover each other exactly.



Notice that these triangles are such that

$AB = PQ, BC = QR, AC = PR, \angle A = \angle P, \angle B = \angle Q$ and $\angle C = \angle R$.

Thus these triangles have the same shape and same size, so they are congruent.

We express this as $\Delta ABC \cong \Delta PQR$.

It means that when we place a trace-copy of ΔABC on ΔPQR, vertex A falls on vertex P, vertex B on vertex Q and vertex C on vertex R. Then side AB falls on PQ, BC on QR and CA on RP. Also $\angle A$ falls on $\angle P$, $\angle B$ on $\angle Q$ and $\angle C$. Thus, the order in which the vertices match, automatically determines a correspondence between the sides and the angles of the two triangles. It follows that if the vertices of ΔABC match the vertices of ΔPQR in the order:

$A \leftrightarrow P, B \leftrightarrow Q, C \leftrightarrow R$

then all the six corresponding parts (3 sides and 3 angles) of two triangles are equal *i.e.*

$AB = PQ, BC = QR, CA = RP, \angle A = \angle P, \angle B = \angle Q$ and $\angle C = \angle R$.

However, if we place ΔABC on ΔPQR such that A falls on Q, then other vertices may not correspond suitably. Take a trace-copy of ΔABC and place vertex A on vertex Q and try to find out!

This shows that while talking about congruence of triangles, not only the measures of angles and the lengths of sides matter, but also the matching of vertices matter. In the above triangles ABC and PQR, the correspondence is

$A \leftrightarrow P, B \leftrightarrow Q, C \leftrightarrow R$.

Remarks

1. Congruent triangles are 'equal in all respects' *i.e.* they are the exact *duplicate* of each other.
2. If two triangles are congruent, then any one can be *superposed* on the other to cover it exactly.
3. In congruent triangles, the sides and the angles which coincide by superposition are called *corresponding sides* and *corresponding angles*.
4. The corresponding sides lie opposite to the equal angles and the corresponding angles lie opposite to the equal sides.

   In the above diagram, $\angle A = \angle P$, therefore, the corresponding sides BC and QR are equal. Also BC = QR, therefore, the corresponding angles A and P are equal.

   The abbreviation 'CPCT' or 'c.p.c.t.' will be used for corresponding parts of congruent triangles.
5. The order of the letters in the names of congruent triangles displays the corresponding relationship between the two triangles.

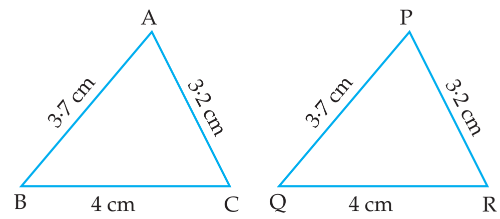
   Thus, when we write $\Delta ABC \cong \Delta PQR$, it means that A lies on P, B lies on Q and C lies on R *i.e.* $\angle A = \angle P, \angle B = \angle Q, \angle C = \angle R$ and $BC = QR, CA = RP, AB = PQ$.

   Writing any other correspondence *i.e.* $\Delta ABC \cong \Delta PRQ$, $\Delta ABC \cong \Delta RPQ$ etc. will be incorrect.

9.2.1 Criteria for congruence of triangles

For any two triangles to be congruent, the *six elements* of one triangle *need not be proved* equal to the corresponding six elements of the other triangle.

In earlier classes, we have learnt that three angles of one triangle equal to three angles of another triangle is not sufficient for the congruence of two triangles. Let us see whether three sides of one triangle equal to three sides of another triangle is enough for the congruence of two triangles.



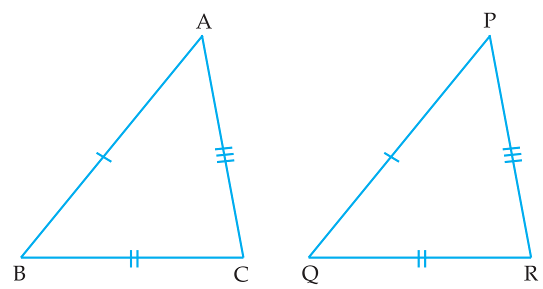
Draw two triangles ABC and PQR such that AB = PQ = 3·7 cm, BC = QR = 4 cm and CA = RP = 3·2 cm.

Make a trace-copy of ΔABC and place it over ΔPQR. We observe that the two triangles cover each other exactly and so they are congruent.

Repeat this activity with more pairs of triangles satisfying these conditions. We observe that the equality of three sides of two triangles is sufficient for the congruence of two triangles. We record it as:

- **SSS congruence rule**

  Two triangles are congruent if the three sides of one triangle are equal to the three sides of the other triangle.

  In the adjoining figure,

  AB = PQ, BC = QR

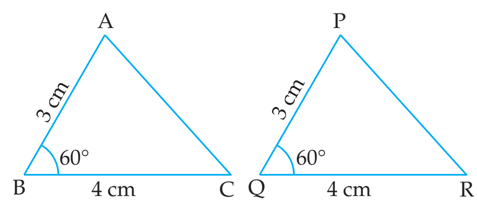
  and AC = PR

  ∴ ΔABC ≅ ΔPQR.

  Note that SSS stands for **Side-Side-Side**.

  Now, let us see whether two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle is enough for the congruence of two triangles.

Draw two triangles ABC and PQR such that BC = QR = 4 cm, ∠B = ∠Q = 60° and AB = PQ = 3 cm.

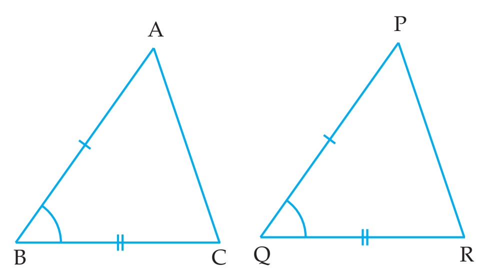


Make a trace-copy of ΔABC and place it over ΔPQR. We observe that the two triangles cover each other exactly and so they are congruent.

Repeat this activity with more pairs of triangles satisfying these conditions. We observe that equality of two sides and the included angle is enough for the congruence of two triangles. We record it as:

- **SAS congruence rule**

  Two triangles are congruent if two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle.

  In the adjoining figure,

  AB = PQ, BC = QR and

  ∠B = ∠Q

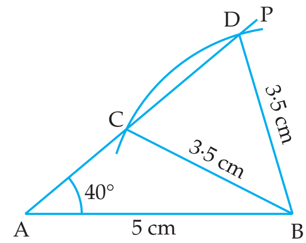
  ∴ ΔABC ≅ ΔPQR

  This is known as **Side-Angle-Side** criterion (or rule) of congruency.

Note. The equality of the 'included angle' is essential.

Look at the adjoining figure:

Here, AB = 5 cm and ∠PAB = 40°. Taking B as centre and radius 3·5 cm, we draw an arc to meet AP at points C and D (as shown in figure).



We consider two triangles ABC and ABD. In these triangles, we have

AB = AB, BC = BD and

∠CAB = ∠DAB (each = 40°)

Thus, two sides and one angle of one triangle are equal to two sides and one angle of the other triangle but the triangles are not congruent which is clear from the figure because ΔABC is a part of ΔABD. In fact, two vertices A and B are same for both triangles but their third vertices C and D do not coincide. Here, note that ∠CAB and ∠DAB are not **included angles.**

Hence, for SAS congruence rule the equality of the included angle is essential.

Thus, SAS congruence rule holds but not ASS or SSA rule.

Illustrative Examples

Example 1. *If ΔPQR ≅ ΔEDF, then is it true to say that PR = EF? Give reason for your answer.*

Solution. Given ΔPQR ≅ ΔEDF. It means that P ↔ E, Q ↔ D and R ↔ F, therefore, PR = EF (corresponding sides are equal).

Hence, the given statement is true.

Example 2. *ABCD is a quadrilateral in which AB = BC and AD = CD. Show that BD bisects both the angles ABC and ADC.*

Solution. Given a quadrilateral ABCD in which AB = BC and AD = CD. Join B and D.

In ΔABD and ΔCBD,

AB = BC (given)

AD = CD (given)

BD = BD (common)

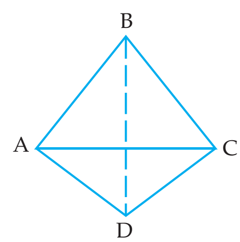
∴ ΔABD ≅ ΔCBD (by SSS congruence rule)

∴ ∠ABD = ∠CBD (c.p.c.t.)

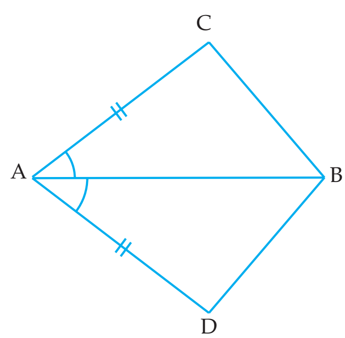
⇒ BD is bisector of ∠ABC.

Also ∠ADB = ∠CDB (c.p.c.t.)

⇒ BD is bisector of ∠ADC.



Example 3. *In the adjoining quadrilateral, AC = AD and AB bisects ∠A. Show that ΔABC ≅ ΔABD. What can you say about BC and BD?*



Solution. As AB bisects ∠A, ∠CAB = ∠DAB.

In ΔABC and ΔABD,

AC = AD (given)

AB = AB (common)

∠CAB = ∠DAB (∵ AB bisects ∠A)

∴ ΔABC ≅ ΔABD

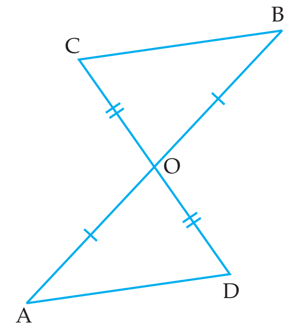
(by SAS rule of congruency)

∴ BC = BD (c.p.c.t.)

Example 4. *In the adjoining figure, OA = OB and OD = OC. Show that*

(i) $\Delta AOD \cong \Delta BOC$

(ii) $AD \parallel BC$.



Solution. (*i*) In ΔAOD and ΔBOC,

OA = OB (given)

OD = OC (given)

∠AOD = ∠BOC (vert. opp. ∠s)

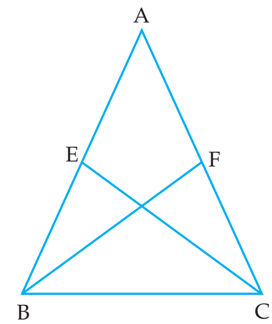
∴ ΔAOD ≅ ΔBOC (by SAS rule of congruency)

(*ii*) ∠OAD = ∠OBC (c.p.c.t.)

But these form a pair of alternate angles for line segments AD and BC.

Therefore, AD ∥ BC.

Example 5. *In the adjoining figure, E and F are respectively the mid-points of equal sides AB and AC of* Δ*ABC. Show that BF = CE.*



Solution. In ΔABF and ΔACE,

AB = AC (given)

AF = AE (halves of equal sides AC and AB)

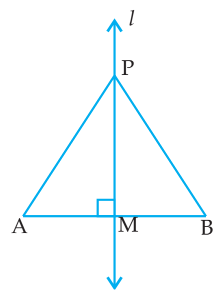
∠A = ∠A (common)

∴ ΔABF ≅ ΔACE (by SAS rule of congruency)

∴ BF = CE (c.p.c.t.)

Example 6. *AB is a line segment and line l is its perpendicular bisector. If P is a point on l, show that P is equidistant from A and B.*

Solution. Given line *l* is perpendicular bisector of AB and P is a point on *l*, so M is mid-point of AB and MP ⊥ AB.



In ΔAMP and ΔBMP,

AM = MB (∵ M is mid-point of AB)

MP = MP (common)

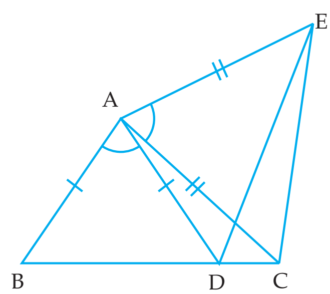
∠AMP = ∠BMP (each = 90°)

∴ ΔAMP ≅ ΔBMP (by SAS rule of congruency)

∴ AP = BP (c.p.c.t.)

⇒ P is equidistant from A and B.

Example 7. *In the adjoining figure, AC = AE, AB = AD and ∠BAD = ∠EAC. Show that BC = DE.*



Solution. Given ∠BAD = ∠EAC

⇒ ∠BAD + ∠DAC = ∠EAC + ∠DAC (adding ∠DAC to both sides)

⇒ ∠BAC = ∠DAE.

In ΔABC and ΔADE,

AB = AD (given)

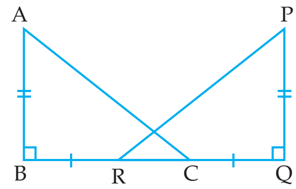
AC = AE (given)

∠BAC = ∠DAE (proved above)

$\therefore \quad \Delta ABC \cong \Delta ADE$ (by SAS rule of congruency)

$\therefore \quad BC = DE$ (c.p.c.t.)

Example 8. *In the adjoining figure, AB = PQ, BR = CQ, AB ⊥ BC and PQ ⊥ RQ. Prove that AC = PR.*



Solution. Given $BR = CQ$

$\Rightarrow \quad BR + RC = CQ + RC$

(adding RC to both sides)

$\Rightarrow \quad BC = QR$

In ΔABC and ΔPQR,

$AB = PQ$ (given)

$BC = QR$ (proved above)

$\angle ABC = \angle PQR$ (each = 90°, $\because$ $AB \perp BC$ and $PQ \perp RQ$)

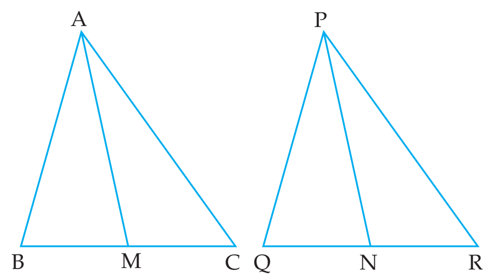
$\therefore \quad \Delta ABC \cong \Delta PQR$ (by SAS rule of congruency)

$\therefore \quad AC = PR$ (c.p.c.t.)

Example 9. *In the adjoining figure, two sides AB, BC and median AM of one triangle ABC are respectively equal to two sides PQ, QR and median PN of triangle PQR. Show that*

(*i*) $\Delta ABM \cong \Delta PQN$

(*ii*) $\Delta ABC \cong \Delta PQR$.



Solution. (*i*) $BC = QR$ (given)

$\Rightarrow \quad \frac{1}{2}BC = \frac{1}{2}QR \Rightarrow BM = QN$

In ΔABM and ΔPQN,

$AB = PQ, BM = QN, AM = PN$

$\therefore \quad \Delta ABM \cong \Delta PQN$ (by SSS rule of congruency)

$\therefore \quad \angle B = \angle Q$ (c.p.c.t.)

(*ii*) In ΔABC and ΔPQR,

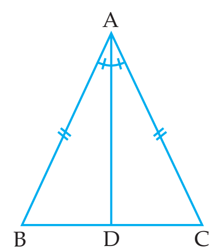
$AB = PQ, BC = QR$ and $\angle B = \angle Q$

$\therefore \quad \Delta ABC \cong \Delta PQR$ (by SAS rule of congruency)

Example 10. *In the figure alongside, AB = AC and AD is bisector of ∠A. Prove that*

(i) D is mid-point of BC

(ii) AD ⊥ BC.



Given. ΔABC, $AB = AC$ and $\angle BAD = \angle CAD$.

To prove. (*i*) $BD = DC$

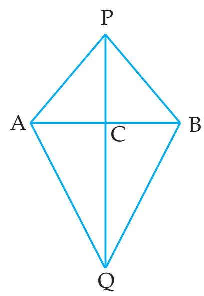
(*ii*) $\angle ADB = 90°$.

Proof.

Statements	*Reasons*
1. AB = AC	1. Given.
2. ∠BAD = ∠CAD	2. AD is bisector of ∠A.
3. AD = AD	3. Common.

4. $\Delta ABD \cong \Delta ACD$	4. SAS rule of congruency
5. (*i*) BD = DC	5. 'c.p.c.t.'
6. $\angle ADB = \angle ADC$	6. 'c.p.c.t.'
7. $\angle ADB + \angle ADC = 180°$	7. BDC is a straight line.
8. (*ii*) $2\angle ADB = 180°$ $\Rightarrow \angle ADB = 90°$. Hence, (*i*) BD = DC and (*ii*) $\angle ADB = 90°$. **Q.E.D.**	8. From 6 and 7.

Example 11. *In the adjoining figure, AB is a line segment. P and Q are points on opposite sides of AB such that each of them is equidistant from the points A and B. Show that the line PQ is the perpendicular bisector of AB.*



Solution. Given AP = BP and AQ = BQ.

We need to show that $PQ \perp AB$ and PQ bisects AB.

In ΔAPQ and ΔBPQ,

AP = BP, AQ = BQ and PQ = PQ (common)

$\therefore$ $\Delta APQ \cong \Delta BPQ$ (by SSS rule of congruency)

$\therefore$ $\angle APC = \angle BPC$ (c.p.c.t.)

In ΔAPC and ΔBPC,

AP = BP, $\angle APC = \angle BPC$ and CP = CP (common)

$\therefore$ $\Delta APC \cong \Delta BPC$ (by SAS rule of congruency)

$\therefore$ AC = CB $\Rightarrow$ C is mid-point of AB

$\Rightarrow$ PQ bisects AB.

Also $\angle ACP = \angle BCP$ (c.p.c.t.)

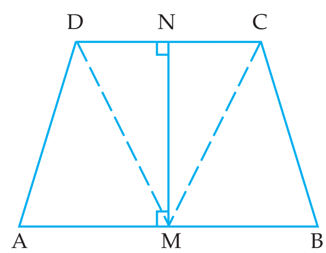
But $\angle ACP + \angle BCP = 180°$ (linear pair)

$\Rightarrow$ $\angle ACP + \angle ACP = 180° \Rightarrow \angle ACP = 90°$

$\Rightarrow$ $PQ \perp AB$.

Example 12. *Line segment joining the mid-points M and N of parallel sides AB and DC respectively of a trapezium ABCD is perpendicular to both the sides AB and DC. Prove that AD = BC.*

Solution. Join CM and DM.



In ΔCMN and ΔDMN,

MN = MN (common)

CN = DN ($\because$ N is mid-point of DC)

$\angle CNM = \angle DNM$

(each = 90°, $\because MN \perp DC$)

$\therefore$ $\Delta CMN \cong \Delta DMN$

(by SAS rule of congruency)

$\therefore$ CM = DM (c.p.c.t.)

and $\angle CMN = \angle DMN$ (c.p.c.t.)

As $MN \perp AB$, $\angle AMN = \angle BMN$ (each = 90°)

$\Rightarrow$ $\angle AMN - \angle DMN = \angle BMN - \angle CMN$ ($\because \angle DMN = \angle CMN$, proved above)

$\Rightarrow$ $\angle AMD = \angle BMC$

In ΔAMD and ΔBMC,

	$AM = BM$	($\because$ M is mid-point of AB)
	$DM = CM$	(proved above)
	$\angle AMD = \angle BMC$	(proved above)
$\therefore$	$\Delta AMD \cong \Delta BMC$	(by SAS rule of congruency)
$\therefore$	$AD = BC$	(c.p.c.t.)

Exercise 9.1

1. It is given that $\Delta ABC \cong \Delta RPQ$. Is it true to say that $BC = QR$? Why?

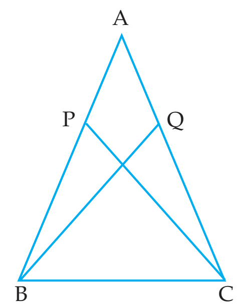
2. "If two sides and an angle of one triangle are equal to two sides and an angle of another triangle, then the two triangles must be congruent." Is the statement true? Why?

3. In the adjoining figure, $AB = AC$ and $AP = AQ$. Prove that

(i) $\Delta APC \cong \Delta AQB$

(ii) $CP = BQ$

(iii) $\angle APC = \angle AQB$.

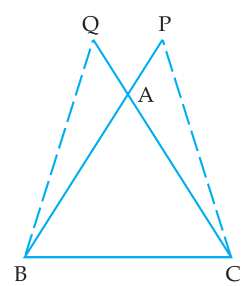


4. In the adjoining figure, $AB = AC$, P and Q are points on BA and CA respectively such that $AP = AQ$. Prove that

(i) $\Delta APC \cong \Delta AQB$

(ii) $CP = BQ$

(iii) $\angle ACP = \angle ABQ$.

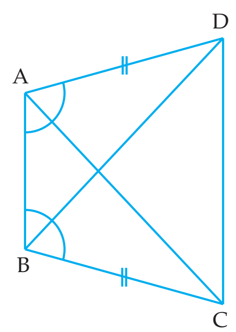


5. In the adjoining figure, ABCD is a quadrilateral in which $AD = BC$ and $\angle DAB = \angle CBA$. Prove that

(i) $\Delta ABD \cong \Delta BAC$

(ii) $BD = AC$

(iii) $\angle ABD = \angle BAC$.



6. In the adjoining figure, $AB = DC$ and $AB \parallel DC$. Prove that $AD = BC$.

Hint: As $AB \parallel DC$, $\angle ABD = \angle CDB$ (alt. $\angle$s). Show that $\Delta ABD \cong \Delta CDB$.