7

INTEGRALS

INTRODUCTION

During the course of study of Mathematics, the reader must have come across several pairs of *inverse operations* like addition — subtraction, multiplication — division, forming an equation whose roots are given — solving a given equation, and so on.

Integration and differentiation are also a pair of inverse operations.

So far, from a given function, we have been finding its derivative but the question arises : "What is the function whose derivative is known"? If the derivative of a function is given, then the function itself is called ***anti-derivative*** or ***integral*** or ***primitive*** of the given function. We illustrate it with the help of an example.

Consider the function $f(x) = x^4$, then its derivative is given by $f'(x) = 4x^3$. The question arises : 'given $f'(x) = 4x^3$, what is $f(x)$?'

However, from our experience of derivatives, we can say that $f(x) = x^4$. This function x^4 is an *anti-derivative* (or *integral* or *primitive*) of $4x^3$.

Yet another question arises : 'is x^4 the only function whose derivative is $4x^3$'? No ! Since the derivative of each of $x^4 + 3$, $x^4 - 5$, $x^4 + 2\sqrt{3}$, $x^4 - 2 + \sqrt{3} + 5\pi$ is $4x^3$, every one of these have an equal claim to be an integral (or anti-derivative) of $4x^3$. Thus, if we know one integral, say x^4, of the given function $4x^3$, then we know many others. In fact, all functions of the form $x^4 + C$ where C is any real number are integrals (or anti-derivatives) of $4x^3$; and there are no others, for, if $f(x)$ and $g(x)$ are two functions such that $f'(x) = g'(x)$ then they differ only by a real constant (number) *i.e.* $g(x) = f(x) + C$ (an application of Lagrange's Mean Value Theorem). This is written as $\int 4x^3\,dx = x^4 + C$. This leads to :

If $\frac{d}{dx}(F(x)) = f(x)$, *then* $\int f(x)\,dx = F(x) + C$.

The function F(x) is called ***anti-derivative*** *or* ***integral*** *or* ***primitive*** *of the function f(x) and C is called* ***constant of integration*** *or* ***arbitrary constant****.*

The function f(x) is called the ***integrand****, x is called the variable of* ***integration,*** *f(x) dx is called the* ***element of integration*** *and the symbol* $\int$ *stands for the integral sign.*

The process of finding functions whose derivative is given is called ***anti-differentiation*** *or* ***integration****.*

Since the integral of a function is not unique *i.e.* definite, therefore, it is called ***indefinite integral***. In integral calculus, we come across two types of problems :

(*i*) the problem of finding a function whose derivative is given.

(*ii*) the problem of finding the area bounded by the graph of a function under given conditions.

These two problems lead to two forms of integrals, ***indefinite integrals*** and ***definite integrals***, which together form the ***integral calculus***.

The two types of integrals *i.e.* indefinite integrals and definite integrals are connected by **fundamental theorem of integral calculus**, which make the definite integrals as a practical tool for sciences, engineering and many other disciplines like economics and probability etc.

7.1 INDEFINITE INTEGRALS

7.1.1 Some elementary standard integrals

We give below some elementary standard integrals which can be obtained directly from our knowledge of derivatives.

1. $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1.$

Proof. Since $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{n+1} \cdot \frac{d}{dx}(x^{n+1}) = \frac{1}{n+1}.(n+1)\, x^n = x^n,$

therefore, by definition of anti-derivative, we have

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1.$$

This is called ***Power Rule***.

Corollary. $\int dx = x + C.$

Proof. $\int dx = \int 1\, dx = \int x^0\, dx = \frac{x^{0+1}}{0+1} + C = x + C.$

2. $\int \frac{1}{x}\, dx = \log |x| + C,\ x \neq 0.$ $\left[\because \frac{d}{dx}(\log|x|) = \frac{1}{x}\right]$

3. $\int a^x\, dx = \frac{a^x}{\log a} + C,\ a > 0,\ a \neq 1.$ $\left[\because \frac{d}{dx}\left(\frac{a^x}{\log a}\right) = \frac{1}{\log a}.\ a^x \log a = a^x\right]$

4. $\int e^x\, dx = e^x + C.$ $\left[\because \frac{d}{dx}(e^x) = e^x\right]$

5. $\int \sin x\, dx = -\cos x + C.$ $\left[\because \frac{d}{dx}(-\cos x) = -(-\sin x) = \sin x\right]$

6. $\int \cos x\, dx = \sin x + C.$ $\left[\because \frac{d}{dx}(\sin x) = \cos x\right]$

7. $\int \sec^2 x\, dx = \tan x + C.$ $\left[\because \frac{d}{dx}(\tan x) = \sec^2 x\right]$

8. $\int \operatorname{cosec}^2 x\, dx = -\cot x + C.$ $\left[\frac{d}{dx}(-\cot x) = -(-\operatorname{cosec}^2 x) = \operatorname{cosec}^2 x\right]$

9. $\int \sec x \tan x\, dx = \sec x + C.$ $\left[\because \frac{d}{dx}(\sec x) = \sec x \tan x\right]$

10. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C.$

$\left[\because \frac{d}{dx}(-\operatorname{cosec} x) = -(-\operatorname{cosec} x \cot x) = \operatorname{cosec} x \cot x\right]$

REMARKS

1. The derivative of a function (if it exists) is unique but the integral of a function is not so. In fact, integrals of a function differ by a real constant (number).
2. We can talk of the derivative of a function at a point but we never talk of the integral of a function at a point. In fact, we always find integral of a function in an interval.

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate the following integrals :*

(i) $\int x^{99}\,dx$ *(ii)* $\int \frac{1}{x^{7/2}}\,dx$ *(iii)* $\int \sqrt[3]{t}\,dt.$

Solution. *(i)* $\int x^{99}\,dx = \frac{x^{99+1}}{99+1} + C = \frac{x^{100}}{100} + C.$

(ii) $\int \frac{1}{x^{7/2}}dx = \int x^{-7/2}\,dx = \frac{x^{-\frac{7}{2}+1}}{-\frac{7}{2}+1} + C = \frac{x^{-\frac{5}{2}}}{-\frac{5}{2}} + C = -\frac{2}{5x^{5/2}} + C.$

(iii) $\int \sqrt[3]{t}\,dt = \int t^{1/3}\,dt = \frac{t^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C = \frac{3}{4}t^{4/3} + C.$

Example 2. *Evaluate the following integrals :*

(i) $\int \frac{\cos x}{1-\cos^2 x}\,dx$ *(ii)* $\int \left(1-2\sin^2\frac{x}{2}\right)dx$ *(iii)* $\int 3^x e^x\,dx.$

Solution. *(i)* $\int \frac{\cos x}{1-\cos^2 x}\,dx = \int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}\,dx$

$= \int \text{cosec } x \cot x\,dx = -\text{cosec } x + C.$

(ii) $\int \left(1-2\sin^2\frac{x}{2}\right)dx = \int \cos x\,dx = \sin x + C.$

(iii) $\int 3^x e^x\,dx = \int (3e)^x\,dx = \frac{(3e)^x}{\log 3e} + C.$

Example 3. *Evaluate the following integrals :*

(i) $\int 9^{\log_3 x}\,dx$ *(ii)* $\int e^{3\log x}(x^4)\,dx.$ (*C.B.S.E. Sample Paper*)

Solution. *(i)* $\int 9^{\log_3 x}\,dx = \int (3^2)^{\log_3 x}\,dx = \int 3^{2\log_3 x}\,dx$

$= \int 3^{\log_3 x^2}\,dx = \int x^2\,dx$ $(\because a^{\log_a x} = x)$

$= \frac{x^3}{3} + C = \frac{1}{3}x^3 + C.$

(ii) $\int e^{3\log x}(x^4)\,dx = \int e^{\log x^3}(x^4)\,dx = \int x^3.x^4\,dx$ $(\because e^{\log x} = x)$

$= \int x^7\,dx = \frac{x^8}{8} + C = \frac{1}{8}x^8 + C.$

Example 4. *Evaluate* $\int a^{\log x}\,dx,\ a > 0,\ x > 0.$

Solution. First we note that $a^{\log x} = x^{\log a}$.

Taking logarithm of both sides, we get

$\log(a^{\log x}) = \log(x^{\log a})$

$\Rightarrow$ $\log x \log a = \log a \log x$, which is true.

$\therefore$ $\int a^{\log x}\,dx = \int x^{\log a}\,dx = \frac{x^{\log a+1}}{\log a+1} + C = \frac{x^{1+\log a}}{1+\log a} + C.$

EXERCISE 7.1

Very short answer type questions

Evaluate the following integrals :

1. *(i)* $\int x^5\,dx$ *(ii)* $\int \frac{1}{x^3}\,dx$ *(iii)* $\int \frac{1}{\sqrt{y}}\,dy.$

2. *(i)* $\int \frac{1}{\sqrt[3]{x}}\,dx$ *(ii)* $\int \sin^2 x \text{ cosec}^2 x\,dx$ *(iii)* $\int \sin x \sec^2 x\,dx.$

3. (*i*) $\int 5^x\,dx$ (*ii*) $\int 7^x\,e^x\,dx$ (*iii*) $\int \frac{4^x}{9^x}\,dx$.

4. (*i*) $\int \frac{\sin x}{1-\sin^2 x}\,dx$ (*ii*) $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}\,dx$ (*iii*) $\int \cos\left(\frac{\pi}{2} - x\right)\,dx$.

(*NCERT*)

5. (*i*) $\int \frac{2}{1-\cos 2x}\,dx$ (*ii*) $\int \sqrt{\frac{1}{2}(1+\cos 2x)}\,dx$.

6. (*i*) $\int e^{2\log x}\,x^{-3}\,dx$ (*ii*) $\int 5^{2\log_5 x}\,dx$ (*iii*) $\int 2^{\log_4 x}\,dx$.

7.1.2 Some properties of indefinite integrals

The following properties help us in reducing a number of problems to elementary standard integrals.

Property 1. $\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x)$.

Proof. Let $\int f(x)\,dx = F(x)$ …(*i*)

Then by definition, $\frac{d}{dx}(F(x)) = f(x)$.

Substituting the value of $F(x)$ from (*i*), we get $\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x)$, as required.

Property 2. $\int k\,f(x)\,dx = k\int f(x)\,dx$, *where k is a constant.*

Proof. Since $\frac{d}{dx}\left(k\int f(x)\,dx\right) = k\,\frac{d}{dx}\left(\int f(x)\,dx\right) = k\,f(x)$ (By property 1)

Therefore, by definition of integral, $\int k\,f(x)\,dx = k\int f(x)\,dx$, as required.

Property 3. $\int (f_1(x) + f_2(x))\,dx = \int f_1(x)\,dx + \int f_2(x)\,dx$.

Proof. Since $\frac{d}{dx}\left(\int (f_1(x)\,dx + \int f_2(x)\,dx\right) = \frac{d}{dx}\left(\int f_1(x)\,dx\right) + \frac{d}{dx}\left(\int f_2(x)\,dx\right)$

$= f_1(x) + f_2(x)$ (By property 1)

Therefore, by definition of integral,

$$\int (f_1(x) + f_2(x))\,dx = \int f_1(x)\,dx + \int f_2(x)\,dx.$$

Property 4. The properties 2 and 3 can be generalised to a finite number of functions $f_1, f_2, \ldots, f_n$ and real numbers $k_1, k_2, \ldots, k_n$ giving

$$\int (k_1 f_1(x) + k_2 f_2(x) + \ldots + k_n f_n(x))\,dx = k_1\int f_1(x)\,dx + k_2\int f_2(x)\,dx + \ldots + k_n\int f_n(x)\,dx.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate the following integrals:*

(i) $\int (3x^5 - 7\sin x + 2)\,dx$ *(ii)* $\int \left(\sqrt{x} - \frac{1}{x^2}\right)^2 dx$.

Solution. (*i*) $\int (3x^5 - 7\sin x + 2)\,dx = 3\int x^5\,dx - 7\int \sin x\,dx + 2\int 1\,dx$

$$= 3.\frac{x^6}{6} - 7.(-\cos x) + 2.x + C = \frac{1}{2}x^6 + 7\cos x + 2x + C.$$

(*ii*) $\int \left(\sqrt{x} - \frac{1}{x^2}\right)^2 dx = \int \left(x + \frac{1}{x^4} - 2.\sqrt{x}.\frac{1}{x^2}\right) dx = \int x\,dx + \int x^{-4}\,dx - 2\int x^{-3/2}\,dx$

$$= \frac{x^2}{2} + \frac{x^{-3}}{-3} - 2.\frac{x^{-1/2}}{-1/2} + C = \frac{x^2}{2} - \frac{1}{3x^3} + \frac{4}{\sqrt{x}} + C.$$

Example 2. *Evaluate the following integrals:*

(i) $\int (2x^2 + 3)(1 - 3x)\, dx$ *(ii)* $\int \frac{(2 + 5t)(3 - t^2)}{\sqrt{t}}\, dt.$

Solution. (*i*) $\int (2x^2 + 3)(1 - 3x)\, dx = \int (2x^2 - 6x^3 + 3 - 9x)\, dx$

$= 2\int x^2\, dx - 6\int x^3\, dx + 3\int 1\, dx - 9\int x\, dx$

$= 2 . \frac{x^3}{3} - 6 . \frac{x^4}{4} + 3x - 9 . \frac{x^2}{2} + C$

$= \frac{2}{3}x^3 - \frac{3}{2}x^4 + 3x - \frac{9}{2}x^2 + C.$

(*ii*) $\int \frac{(2 + 5t)(3 - t^2)}{\sqrt{t}}\, dt = \int \frac{6 + 15t - 2t^2 - 5t^3}{\sqrt{t}}\, dt$

$= \int \left(\frac{6}{\sqrt{t}} + 15\sqrt{t} - 2t^{3/2} - 5t^{5/2}\right) dt$

$= 6\int t^{-1/2}\, dt + 15\int t^{1/2}\, dt - 2\int t^{3/2}\, dt - 5\int t^{5/2}\, dt$

$= 6 . \frac{t^{1/2}}{1/2} + 15 . \frac{t^{3/2}}{3/2} - 2 . \frac{t^{5/2}}{5/2} - 5 . \frac{t^{7/2}}{7/2} + C$

$= 12\sqrt{t} + 10t^{3/2} - \frac{4}{5}t^{5/2} - \frac{10}{7}t^{7/2} + C.$

Example 3. *Evaluate:*

(i) $\int cosec\ x\ (cosec\ x + cot\ x)\, dx$ *(NCERT)* *(ii)* $\int (e^{x \log a} + e^{a \log x} - e^{a \log a})\, dx.$

Solution. (*i*) $\int \text{cosec}\ x\ (\text{cosec}\ x + \cot x)\, dx = \int (\text{cosec}^2 x + \text{cosec}\ x \cot x)\, dx$

$= \int \text{cosec}^2 x\, dx + \int \text{cosec}\ x \cot x\, dx = -\cot x - \text{cosec}\ x + C.$

(*ii*) $\int (e^{x \log a} + e^{a \log x} - e^{a \log a})\, dx = \int ((e^{\log a})^x + (e^{\log x})^a - (e^{\log a})^a)\, dx$

$= \int (a^x + x^a - a^a)\, dx$ $(\because\ e^{\log x} = x)$

$= \frac{a^x}{\log a} + \frac{x^{a+1}}{a+1} - a^a . x + C.$

Example 4. *Evaluate the following integrals:*

(i) $\int \left(x - \frac{1}{x}\right)^3 dx$ *(ii)* $\int \frac{(a^x + b^x)^2}{a^x b^x}\, dx.$

Solution. (*i*) $\int \left(x - \frac{1}{x}\right)^3 dx = \int \left(x^3 - \frac{1}{x^3} - 3x . \frac{1}{x}\left(x - \frac{1}{x}\right)\right) dx$

$= \int \left(x^3 - x^{-3} - 3x + 3 . \frac{1}{x}\right) dx$

$= \frac{x^4}{4} - \frac{x^{-2}}{-2} - 3 . \frac{x^2}{2} + 3 \log |x| + C$

$= \frac{x^4}{4} + \frac{1}{2x^2} - \frac{3}{2}x^2 + 3 \log |x| + C.$

(*ii*) $\int \frac{(a^x + b^x)^2}{a^x b^x}\, dx = \int \frac{a^{2x} + b^{2x} + 2a^x b^x}{a^x b^x}\, dx = \int \left(\left(\frac{a}{b}\right)^x + \left(\frac{b}{a}\right)^x + 2\right) dx$

$= \frac{\left(\frac{a}{b}\right)^x}{\log \frac{a}{b}} + \frac{\left(\frac{b}{a}\right)^x}{\log \frac{b}{a}} + 2x + C,\ a \neq b.$

In particular, if $a = b$, then

$\int \frac{(a^x + b^x)^2}{a^x b^x}\, dx = \int \frac{(2a^x)^2}{a^x . a^x}\, dx = \int 4\, dx = 4x + C.$

Example 5. *Evaluate the following integrals:*

(i) $\int tan^2\ x\ dx$ *(ii)* $\int \sqrt{1-sin\,2x}\ dx.$

Solution. (*i*) $\int \tan^2 x\ dx = \int (\sec^2 x - 1)\ dx = \tan x - x + C.$

(*ii*) $\int \sqrt{1-\sin 2x}\ dx = \int \sqrt{\cos^2 x + \sin^2 x - 2\sin x \cos x}\ dx$

$= \int (\cos x - \sin x)\ dx = \sin x - (-\cos x) + C = \sin x + \cos x + C.$

Example 6. *Evaluate the following integrals:*

(i) $\int \frac{1}{1+sec\ x}\ dx$ *(ii)* $\int \frac{1+sin\ x}{1-sin\ x}\ dx$ (*C.B.S.E. 2000 C*)

Solution. (*i*) $\int \frac{1}{1+\sec x} dx = \int \frac{1}{\sec x + 1} \times \frac{\sec x - 1}{\sec x - 1} dx = \int \frac{\sec x - 1}{\sec^2 x - 1}\ dx$

$= \int \frac{\sec x - 1}{\tan^2 x} dx = \int \left(\frac{\sec x}{\tan^2 x} - \frac{1}{\tan^2 x}\right) dx$

$= \int (\text{cosec}\ x \cot x - \cot^2 x)\ dx = \int \text{cosec}\ x \cot x\ dx - \int (\text{cosec}^2 x - 1)\ dx$

$= -\ \text{cosec}\ x - (-\cot x - x) + C = -\ \text{cosec}\ x + \cot x + x + C.$

(*ii*) $\int \frac{1+\sin x}{1-\sin x}\ dx = \int \frac{1+\sin x}{1-\sin x} \times \frac{1+\sin x}{1+\sin x} dx = \int \frac{(1+\sin x)^2}{1-\sin^2 x}\ dx$

$= \int \frac{1+2\sin x+\sin^2 x}{\cos^2 x}\ dx = \int (\sec^2 x + 2\sec x \tan x + \tan^2 x)\ dx$

$= \int (\sec^2 x + 2\sec x \tan x + \sec^2 x - 1)\ dx$

$= \int (2\sec^2 x + 2\sec x \tan x - 1)\ dx$

$= 2\tan x + 2\sec x - x + C.$

Example 7. *Evaluate the following integrals:*

(i) $\int cos^{-1}\ (sin\ x)\ dx$ *(ii)* $\int tan^{-1}\ (sec\ x + tan\ x)\ dx$

(iii) $\int tan^{-1} \left(\sqrt{\frac{1-sin\ x}{1+sin\ x}}\right) dx.$ (*C.B.S.E. 2006 C, 03*)

Solution. (*i*) $\int \cos^{-1}(\sin x)\ dx = \int \cos^{-1}\left(\cos\left(\frac{\pi}{2} - x\right)\right) dx = \int \left(\frac{\pi}{2} - x\right) dx$

$= \frac{\pi}{2}.x - \frac{x^2}{2} + C.$

(*ii*) $\int \tan^{-1}(\sec x + \tan x)\ dx = \int \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right) dx$

$= \int \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right) dx = \int \tan^{-1}\left(\frac{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}\right) dx$

$= \int \tan^{-1}\left(\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right) dx = \int \tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right) dx = \int \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right) dx$

$= \int \left(\frac{\pi}{4} + \frac{x}{2}\right) dx = \frac{\pi}{4}.x + \frac{1}{2}.\frac{x^2}{2} + C = \frac{x}{4}\ (\pi + x) + C.$

$$(iii)\ \int \tan^{-1}\left(\sqrt{\frac{1-\sin x}{1+\sin x}}\right) dx = \int \tan^{-1}\left(\sqrt{\frac{1-\cos\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}}\right) dx = \int \tan^{-1}\left(\sqrt{\frac{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}}\right) dx$$

$$= \int \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right) dx = \int \left(\frac{\pi}{4}-\frac{x}{2}\right) dx$$

$$= \frac{\pi}{4}\,.\,x - \frac{1}{2}\cdot\frac{x^2}{2} + C = \frac{\pi}{4}x - \frac{x^2}{4} + C.$$

Example 8. *Evaluate the following integrals :*

(i) $\int \frac{dx}{\sin^2 x\ \cos^2 x}$ *(NCERT)* *(ii)* $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}\, dx.$ *(C.B.S.E. 2013)*

Solution. *(i)* $\int \frac{dx}{\sin^2 x\ \cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x\ \cos^2 x}\, dx = \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right) dx$

$$= \int \sec^2 x\, dx + \int \text{cosec}^2\, x\, dx = \tan x - \cot x + C.$$

(ii) $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}\, dx = \int \frac{(2\cos^2 x - 1) - (2\cos^2\alpha - 1)}{\cos x - \cos\alpha}\, dx$

$$= \int \frac{2(\cos^2 x - \cos^2\alpha)}{\cos x - \cos\alpha}\, dx = 2\int (\cos x + \cos\alpha)\, dx$$

$$= 2\int \cos x\, dx + 2\cos\alpha \int 1\, dx = 2\sin x + 2\cos\alpha\,.\,x + C$$

$$= 2(\sin x + x\cos\alpha) + C.$$

Example 9. *If* $f'(x) = 4x^3 - \frac{3}{x^4}$ *and* $f(2) = 0$, *find* $f(x)$. *(NCERT)*

Solution. Given $f'(x) = 4x^3 - \frac{3}{x^4}$,

$$\therefore \quad \text{by def; } f(x) = \int \left(4x^3 - \frac{3}{x^4}\right) dx = \int (4x^3 - 3x^{-4})\, dx$$

$$= 4\,.\,\frac{x^4}{4} - 3\,.\,\frac{x^{-3}}{-3} + C = x^4 + \frac{1}{x^3} + C.$$

But $f(2) = 0 \Rightarrow 2^4 + \frac{1}{2^3} + C = 0 \Rightarrow 16 + \frac{1}{8} + C = 0$

$$\Rightarrow \qquad C = -16 - \frac{1}{8} = -\frac{129}{8}.$$

$$\therefore \qquad f(x) = x^4 + \frac{1}{x^3} - \frac{129}{8}.$$

EXERCISE 7.2

Very short answer type questions (1 to 17) :

Evaluate the following (1 to 30) integrals :

1. *(i)* $\int (ax^2 + bx + c)\, dx$ *(NCERT)* *(ii)* $\int (x^{2/3} + 1)\, dx.$ *(NCERT)*

2. *(i)* $\int\left(\frac{1}{x^2} + \sec^2 x + 5x\right) dx$ *(ii)* $\int \left(\frac{2a}{\sqrt{x}} - \frac{b}{x^2} + 3c\sqrt[3]{x^2}\right) dx.$ *(NCERT Examplar Problems)*

3. *(i)* $\int \frac{2\cos x}{3\sin^2 x}\, dx$ *(C.B.S.E. 2008)* *(ii)* $\int (2x^2 - 3\sin x + 5\sqrt{x})\, dx.$ *(NCERT)*

4. *(i)* $\int (2x - 3\cos x + e^x)\, dx$ *(NCERT)* *(ii)* $\int x^2\left(1 - \frac{1}{x^2}\right) dx.$ *(NCERT)*

5. (*i*) $\int \left(x^{3/2} + 2e^x - \frac{1}{x}\right) dx$ (*NCERT*) (*ii*) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$. (*NCERT*)

6. (*i*) $\int (1 - x)\sqrt{x}\ dx$ (*C.B.S.E. 2012*) (*ii*) $\int \sqrt{x}\,(3x^2 + 2x + 3)\ dx$. (*NCERT*)

7. (*i*) $\int \frac{x^3 + 5x^2 - 4}{x^2}\ dx$ (*NCERT*) (*ii*) $\int \frac{x^3 + 3x + 4}{\sqrt{x}}\ dx$. (*NCERT*)

8. (*i*) $\int (2x - 1)(x^2 + 1)\ dx$ (*ii*) $\int x^2 \left(2x - \frac{1}{x}\right)^2 dx$.

9. (*i*) $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$ (*ii*) $\int (x + \sqrt{x})^3\ dx$.

10. (*i*) $\int \frac{(x + a)^2}{\sqrt{x}}\ dx$ (*ii*) $\int \frac{x^3 - x^2 + x - 1}{x - 1}\ dx$. (*NCERT*)

11. (*i*) $\int \frac{1 - \sin x}{\cos^2 x}\ dx$ (*NCERT*) (*ii*) $\int \frac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}}\ dx$. (*NCERT Examplar Problems*)

12. (*i*) $\int \cot^2 x\ dx$ (*ii*) $\int \left(\tan^2 x - 7x^2 + \frac{1}{x^4}\right) dx$.

13. (*i*) $\int \sec x\,(\sec x + \tan x)\ dx$ (*C.B.S.E. 2011*) (*ii*) $\int \frac{\sec^2 x}{\text{cosec}^2 x}\ dx$. (*C.B.S.E. 2011*)

14. (*i*) $\int \frac{\sin 2x}{\sin x}\ dx$ (*ii*) $\int \frac{1}{1 - \cos 2x}\ dx$.

15. (*i*) $\int \left(\frac{\cot x}{\sin x} - \frac{\tan x}{\cos x} - 3\tan^2 x\right) dx$ (*ii*) $\int \frac{2 - 3\sin x}{\cos^2 x}\ dx$. (*C.B.S.E. 2011*)

16. (*i*) $\int \frac{1}{1 + \cos x}\ dx$ (*NCERT Examplar Problems*) (*ii*) $\int \frac{1}{1 - \sin x}\ dx$. (*C.B.S.E. 2002 C*)

17. (*i*) $\int \frac{1}{1 + \text{cosec}\, x}\ dx$ (*ii*) $\int \sin^{-1}(\cos x)\ dx$. (*NCERT*)

18. (*i*) $\int \frac{\sin x}{1 - \sin x}\ dx$ (*ii*) $\int \frac{1 - \sin x}{1 + \sin x}\ dx$.

19. (*i*) $\int \frac{\sin^2 x}{1 + \cos x}\ dx$ (*NCERT*) (*ii*) $\int \sqrt{1 + \sin 2x}\ dx$.

20. (*i*) $\int (2\tan x - 3\cot x)^2\ dx$ (*ii*) $\int \left(x^{3/2} - \cos^2 \frac{x}{2}\right) dx$.

21. (*i*) $\int \left(\sqrt{x} - \sin \frac{x}{2}\cos \frac{x}{2} + 5\right) dx$ (*ii*) $\int \frac{\cos 2x}{\sin^2 x \cos^2 x}\ dx$.

22. (*i*) $\int \frac{\sec x}{\sec x + \tan x}\ dx$ (*ii*) $\int \frac{\tan x}{\sec x + \tan x}\ dx$.

23. (*i*) $\int \frac{\cos x - \cos 2x}{1 - \cos x}\ dx$ (*NCERT Examplar Problems*) (*ii*) $\int \frac{x^6 + 1}{x^2 + 1}\ dx$.

24. (*i*) $\int \sqrt{1 + \cos 2x}\ dx$ (*ii*) $\int \sqrt{1 - \cos 2x}\ dx$.

25. (*i*) $\int \frac{1-\cos 2x}{1+\cos 2x}\,dx$ (*ii*) $\int \frac{\sin^3 x+\cos^3 x}{\sin^2 x\,\cos^2 x}\,dx.$ (*NCERT*)

26. (*i*) $\int \frac{\cot x}{\text{cosec } x-\cot x}\,dx$ (*ii*) $\int \frac{\text{cosec } x}{\text{cosec } x-\cot x}\,dx.$

27. (*i*) $\int \frac{5\cos^3 x+6\sin^3 x}{\sin^2 x\,\cos^2 x}\,dx$ (*ii*) $\int \frac{\sin x+\cos x}{\sqrt{1+\sin 2x}}\,dx.$

(*NCERT Examplar Problems*)

28. (*i*) $\int \cot^{-1}\left(\frac{\sin 2x}{1-\cos 2x}\right)dx$ (*ii*) $\int \tan^{-1}\left(\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right)dx.$

(*C.B.S.E. 2003*)

29. (*i*) $\int \sin^{-1}\left(\frac{2\tan x}{1+\tan^2 x}\right)dx$ (*ii*) $\int \cos^{-1}\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)dx.$

30. (*i*) $\int \tan^{-1}(\text{cosec } x-\cot x)\,dx$ (*ii*) $\int \tan^{-1}(\text{cosec } x+\cot x)\,dx.$

31. If $f'(x) = 4x^3 - 6$ and $f(0) = 3$, find $f(x)$. (*NCERT*)

32. If $f'(x) = a\sin x + b\cos x$, $f'(0) = 4$, $f(0) = 3$ and $f\left(\frac{\pi}{2}\right) = 5$, find $f(x)$.

7.2 INTEGRATION BY SUBSTITUTION

Some functions can be integrated directly by the use of standard integrals while there exist some functions which cannot be integrated directly but can be reduced to the standard integrals by proper substitution *i.e.* by the introduction of a new variable. The method of evaluating an integral by reducing it to standard form by a substitution is called **integration by substitution.**

A special form

If $\int f(x)\,dx = F(x) + C$, then

$$\int f(ax+b)\,dx = \frac{F(ax+b)}{a} + C,\ a \neq 0.$$

Proof. Put $ax + b = t$, then $a\,dx = dt \Rightarrow dx = \frac{1}{a}\,dt$,

$$\therefore \quad \int f(ax+b)\,dx = \int f(t)\,.\,\frac{1}{a}\,dt = \frac{1}{a}\int f(t)\,dt$$

$$= \frac{1}{a}F(t) + C = \frac{F(ax+b)}{a} + C,\ a \neq 0.$$

Hence, we have the following important rule:

If the integral of a function of x is known, then if x is multiplied by a constant and to the product is added another constant, the integral is of the same form but it is divided by the coefficient of x.

REMARK

Long Division

In a rational function, if the degree of the numerator of the integrand is equal to or greater than that of the denominator, divide the numerator by the denominator until the degree of the remainder is less than that of the denominator.

In this context, remember that

$$\frac{\text{numerator}}{\text{denominator}} = \text{quotient} + \frac{\text{remainder}}{\text{denominator}}.$$

47. (*i*) $\int_0^{\pi/2} \sqrt{\sin x}\ \cos^5 x\ dx$ (*NCERT*) (*ii*) $\int_0^{\pi/4} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\, dx.$ (*NCERT*)

By using properties of definite integrals, evaluate the following (48 to 51) :

48. $\int_0^4 (|x| + |x-2| + |x-4|)\, dx$ (*C.B.S.E. 2013*)

49. (*i*) $\int_{1/e}^{e} |\log x|\, dx$ (*ii*) $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{1+\cot x}}\, dx.$

Hint. (*i*) $|\log x| = \begin{cases} -\log x, \text{if } \frac{1}{e} \le x \le 1 \\ \log x, \text{if } 1 \le x \le e. \end{cases}$

50. (*i*) $\int_0^1 x(1-x)^5\, dx$ (*ii*) $\int_0^{\pi/2} \frac{\sin^2 x}{1+\sin x \cos x}\, dx.$

51. (*i*) $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx$ (*ii*) $\int_0^{\pi/2} \frac{a \sec x + b \operatorname{cosec} x}{\sec x + \operatorname{cosec} x}\, dx.$

Hint. (*i*) $\tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) = \tan^{-1}\left(\frac{x+(x-1)}{1-x(x-1)}\right)$

$= \tan^{-1} x + \tan^{-1}(x-1).$

Let $I = \int_0^1 (\tan^{-1} x + \tan^{-1}(x-1))\, dx = \int_0^1 (\tan^{-1}(1-x) + \tan^{-1}(-x))\, dx$ (by property P_4)

$= -I.$

ANSWERS

EXERCISE 7.1

1. (*i*) $\frac{x^6}{6} + C$ (*ii*) $-\frac{1}{2x^2} + C$ (*iii*) $2\sqrt{y} + C$

2. (*i*) $\frac{3}{2} x^{2/3} + C$ (*ii*) $x + C$ (*iii*) $\sec x + C$

3. (*i*) $\frac{5^x}{\log 5} + C$ (*ii*) $\frac{(7e)^x}{\log 7e} + C$ (*iii*) $\frac{\left(\frac{4}{9}\right)^x}{\log \frac{4}{9}} + C$

4. (*i*) $\sec x + C$ (*ii*) $\tan x + C$ (*iii*) $-\cos x + C$

5. (*i*) $-\cot x + C$ (*ii*) $\sin x + C$

6. (*i*) $\log x + C$ (*ii*) $\frac{1}{3}x^3 + C$ (*iii*) $\frac{2}{3}x^{3/2} + C$

EXERCISE 7.2

1. (*i*) $\frac{ax^3}{3} + \frac{bx^2}{2} + cx + C$ (*ii*) $\frac{3}{5} x^{5/3} + x + C$

2. (*i*) $-\frac{1}{x} + \tan x + \frac{5}{2} x^2 + C$ (*ii*) $4a\sqrt{x} + \frac{b}{x} + \frac{9c}{5} x^{\frac{5}{3}} + C$

3. (*i*) $-\frac{2}{3} \operatorname{cosec} x + C$ (*ii*) $\frac{2}{3}x^3 + 3\cos x + \frac{10}{3} x^{3/2} + C$

4. (*i*) $x^2 - 3\sin x + e^x + C$ (*ii*) $\frac{1}{3}x^3 - x + C$