DETERMINANTS

4

INTRODUCTION

In 1693, Leibnitz developed determinants to solve a system of equations quickly. We know that a system of algebraic equations like

$$a_1x + b_1 y = c_1 \text{ and } a_2x + b_2 y = c_2$$

can be represented as $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

Further, the above system of equations has a unique solution or not is determined by the number $a_1b_2 - a_2b_1$. The number $a_1b_2 - a_2b_1$ which determines the uniqueness of the solution is associated with the matrix A = $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and is called determinant of A. Determinants have wide applications in Engineering, Sciences, Social Sciences and Economics etc.

In the present chapter, we shall learn about determinants, their expansions, minors and cofactors, their elementary properties, application of determinants in finding the area of a triangle, adjoint and inverse of a square matrix, consistency and inconsistency of system of linear equations and unique solution of linear equations in two or three variables using inverse of a matrix.

4.1 DETERMINANTS

To every square matrix A = $[a_{ij}]$ of order n, $a_{ij} \in \mathbf{R}$, we can associate a unique real number called **determinant of matrix** A, denoted by det A or $|A|$. This may be thought as a function which associates each square matrix with a unique real number *i.e.* $f : M \to \mathbf{R}$ given by $f(A) = \det A = |A|$, where M is the set of square matrices and **R** is the set of real numbers. $|A|$ or det A is read as determinant of A.

If A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant is written as det A = $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Sometimes, a determinant is denoted by Δ. Note that only square matrices have determinants.

4.1.1 Value of a determinant

Determinant of a matrix of order one

Let A = $[a]$ be a square matrix of order 1, then det A = $|a|$ and the value of the determinant is the number itself *i.e.* $|a| = a$.

For example,

(*i*) if A = [5], then det A = | 5 | = 5

(*ii*) if A = [– 7], then det A = | –7 | = –7.

REMARK

A determinant of order one should not be confused with the absolute value of a real (or complex) number.

Determinant of a matrix of order two

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a square matrix of order 2, then $\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ and its value is given by

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\,|\,a_{22}\,| - a_{12}\,|\,a_{21}\,| = a_{11}\,a_{22} - a_{12}\,a_{21} \qquad \ldots(i)$$

The numbers a_{11}, a_{12}, a_{21}, a_{22} are called the elements of the determinant. Note that a determinant of order 2 contains 4 elements. It has two rows and two columns. The expression on the R.H.S. of (*i*) is called the **expansion** (or **value**) of the determinant.

For example,

(*i*) if $A = \begin{bmatrix} 5 & 3 \\ 2 & -1 \end{bmatrix}$, then $\det A = \begin{vmatrix} 5 & 3 \\ 2 & -1 \end{vmatrix} = 5\,.\,(-1) - 3\,.\,2 = -5 - 6 = -11.$

(*ii*) if $A = \begin{bmatrix} 2x & 3y \\ 7x^2 & -5y \end{bmatrix}$, then

$$\det A = \begin{vmatrix} 2x & 3y \\ 7x^2 & -5y \end{vmatrix} = 2x\,.\,(-5y) - 3y\,.\,7x^2 = -10\,xy - 21\,x^2\,y.$$

Determinant of a matrix of order three

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3, then

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{ and its value is given by}$$

$$\det A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}\,a_{33} - a_{23}\,a_{32}) - a_{12}(a_{21}\,a_{33} - a_{23}\,a_{31}) + a_{13}(a_{21}\,a_{32} - a_{22}\,a_{31})$$

$$= a_{11}\,a_{22}\,a_{33} - a_{11}\,a_{23}\,a_{32} - a_{12}\,a_{21}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{21}\,a_{32} - a_{13}\,a_{22}\,a_{31} \qquad \ldots(ii)$$

The numbers a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33} are called the **elements** of the determinant. Note that a determinant of order 3 contains 3^2 *i.e.* 9 elements. The elements a_{11}, a_{12}, a_{13}; a_{21}, a_{22}, a_{23} and a_{31}, a_{32}, a_{33} constitute the **first, second** and **third rows** respectively and the elements a_{11}, a_{21}, a_{31}; a_{12}, a_{22}, a_{32} and a_{13}, a_{23}, a_{33} constitute the **first, second** and **third columns** respectively. The elements a_{11}, a_{22}, a_{33} are called the **diagonal elements** and the line containing these elements is called the **principal diagonal** of the determinant.

The expression on R.H.S. of (ii) is called the ***expansion of the determinant*** *by the first row.*

WORKING RULE

(*i*) Write the elements of the first row with alternatively positive and negative sign, the first element always has positive sign before it.

(*ii*) Multiply each signed element by the determinant of second order obtained after deleting the row and the column in which that element occurs.

For example,

$$\begin{vmatrix} 3 & -2 & 5 \\ 1 & 2 & -1 \\ 0 & 4 & 7 \end{vmatrix} = 3\begin{vmatrix} 2 & -1 \\ 4 & 7 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -1 \\ 0 & 7 \end{vmatrix} + 5\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix}$$

$$= 3(2.7 - 4(-1)) + 2(1.7 - 0.(-1)) + 5(1.4 - 0.2)$$

$$= 3(14 + 4) + 2(7 - 0) + 5(4 - 0)$$

$$= 3.18 + 2.7 + 5.4 = 54 + 14 + 20 = 88.$$

Determinant of a matrix of order four and of higher order

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix}$ be a square matrix of order 4, then det A = $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$.

Note that det A contains 4^2 *i.e.* 16 elements and its expansion (or value) can be obtained in a manner similar to that of a determinant of order 3.

Similarly, we can define determinants of order 5 and of higher orders. However, in this chapter, we shall be mainly dealing with determinants of order ≤ 3.

4.1.2 Minors and cofactors

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}$

be a determinant of order n, n ≥ 2, then the determinant of order n – 1 obtained from the determinant Δ after deleting the ith row and jth column is called the ***minor of the element*** $\mathbf{a_{ij}}$ *and it is, usually, denoted by* M_{ij} *where i = 1, 2, 3, …, n and j = 1, 2, 3, … , n.*

If M_{ij} *is the minor of the element* a_{ij} *in the determinant Δ, then the number* $(-1)^{i+j} M_{ij}$ *is called the* ***cofactor of the element*** $\mathbf{a_{ij}}$, *it is usually denoted by* A_{ij}.

Thus $A_{ij} = (-1)^{i+j} M_{ij}$.

Note that $A_{ij} = M_{ij}$ if $i + j$ is even and

$A_{ij} = -M_{ij}$ if $i + j$ is odd.

For example,

1. Let $\Delta = \begin{vmatrix} 2 & -3 \\ 4 & 7 \end{vmatrix}$, then

$M_{11} = |7| = 7,\ M_{12} = |4| = 4,$

$M_{21} = |-3| = -3,\ M_{22} = |2| = 2$ and

$A_{11} = (-1)^{1+1} M_{11} = 7,\ A_{12} = (-1)^{1+2} M_{12} = -4,$

$A_{21} = (-1)^{2+1} M_{21} = -(-3) = 3,\ A_{22} = (-1)^{2+2} M_{22} = 2.$

2. Let $\Delta = \begin{vmatrix} 7 & 4 & -1 \\ 2 & 3 & 0 \\ 1 & -5 & 2 \end{vmatrix}$, then

$$M_{11} = \begin{vmatrix} 3 & 0 \\ -5 & 2 \end{vmatrix} = 3.2 - (-5).0 = 6,$$

$$M_{22} = \begin{vmatrix} 7 & -1 \\ 1 & 2 \end{vmatrix} = 7.2 - 1.(-1) = 15,$$

$$M_{32} = \begin{vmatrix} 7 & -1 \\ 2 & 0 \end{vmatrix} = 7.0 - 2.(-1) = 2 \text{ etc.}$$

$$A_{11} = (-1)^{1+1} M_{11} = 6, \ A_{22} = (-1)^{2+2} M_{22} = 15 \text{ and}$$

$$A_{32} = (-1)^{3+2} M_{32} = -2 \text{ etc.}$$

For quick working, the signs of the different cofactors according to the positions of the corresponding elements in determinants of order 2 and 3 are given by

$$\begin{vmatrix} + & - \\ - & + \end{vmatrix}, \ \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}.$$

Expansion of a determinant by any row or any column

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ be a determinant of order 3, then

$$A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \ A_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \ A_{21} = (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix},$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \ A_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix},$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \ A_{32} = (-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

and $A_{33} = (-1)^{3+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$.

We know that the value of the determinant Δ is given by

$$\Delta = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(Expansion by first row)

$$= a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$$

(By using values of the cofactors A_{11}, A_{12}, A_{13})

Similarly, we can show that

$$\Delta = a_{21} A_{21} + a_{22} A_{22} + a_{23} A_{23},$$

$$\Delta = a_{11} A_{11} + a_{21} A_{21} + a_{31} A_{31} \text{ etc.}$$

Thus, we have:

The sum of the products of elements of any row (or column) of a determinant with their corresponding cofactors is equal to the value of the determinant.

The above result is true for every determinant of order ≥ 2.

Also, ***it follows that the value of a determinant can be obtained by expanding it with any row or any column.***

REMARK

We can obtain the value of a determinant very quickly if we expand it with the help of a row or a column which contains the maximum number of zeros.

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate the following determinants :*

(i) $\begin{vmatrix} -3 & 1 \\ 5 & 6 \end{vmatrix}$ (NCERT) (ii) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ (NCERT) (iii) $\begin{vmatrix} \cos 15° & \sin 15° \\ \sin 75° & \cos 75° \end{vmatrix}$ (C.B.S.E. 2011)

Solution. (i) $\begin{vmatrix} -3 & 1 \\ 5 & 6 \end{vmatrix} = (-3).6 - 5.1 = -18 - 5 = -23.$ (Expanding by C_1)

(ii) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos\theta \times \cos\theta - \sin\theta\,(-\sin\theta) = \cos^2\theta + \sin^2\theta = 1.$

(iii) $\begin{vmatrix} \cos 15° & \sin 15° \\ \sin 75° & \cos 75° \end{vmatrix} = \cos 15° \times \cos 75° - \sin 15° \times \sin 75°$

$= \cos(15° + 75°)$ $(\because \cos x \cos y - \sin x \sin y = \cos(x + y))$

$= \cos 90° = 0.$

Example 2. *If* $\begin{vmatrix} x-2 & -3 \\ 3x & 2x \end{vmatrix} = 3$, *find the integral value(s) of x.*

Solution. Given $\begin{vmatrix} x-2 & -3 \\ 3x & 2x \end{vmatrix} = 3$ (Expanding by C_1)

$\Rightarrow (x-2).2x - 3x(-3) = 3$

$\Rightarrow 2x^2 - 4x + 9x - 3 = 0 \Rightarrow 2x^2 + 5x - 3 = 0$

$\Rightarrow (2x-1)(x+3) = 0 \Rightarrow 2x - 1 = 0$ or $x + 3 = 0$

$\Rightarrow x = \frac{1}{2}$ or -3 but x is an integer

$\Rightarrow x = -3.$

Example 3. *What positive value of x makes the following pair of determinants equal?*

$\begin{vmatrix} 2x & 3 \\ 5 & x \end{vmatrix}, \begin{vmatrix} 16 & 3 \\ 5 & 2 \end{vmatrix}.$ (C.B.S.E. 2010)

Solution. Given $\begin{vmatrix} 2x & 3 \\ 5 & x \end{vmatrix} = \begin{vmatrix} 16 & 3 \\ 5 & 2 \end{vmatrix}$

$\Rightarrow 2x \times x - 5 \times 3 = 16 \times 2 - 5 \times 3$

$\Rightarrow 2x^2 - 15 = 32 - 15 \Rightarrow 2x^2 = 32 \Rightarrow x^2 = 16$

$\Rightarrow x = 4, -4$ but $x > 0$

$\Rightarrow x = 4.$

Example 4. *Let* $\begin{vmatrix} 3 & y \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$. *Find the possible values of x and y if x, y are natural numbers.*

Solution. Given $\begin{vmatrix} 3 & y \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$

$\Rightarrow \quad 3 \times 1 - x \times y = 3 \times 1 - 4 \times 2$

$\Rightarrow \quad 3 - xy = 3 - 8 \quad \Rightarrow xy = 8.$

If $\quad x = 1, y = 8; \quad x = 2, y = 4;$

$\quad x = 4, y = 2; \quad x = 8, y = 1.$

Example 5. *What is the value of the determinant* $\begin{vmatrix} 0 & 2 & 0 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix}$? *(C.B.S.E. 2010)*

Solution. As the first row contains two zeros, expanding the given determinant by the first row, we get

$$\begin{vmatrix} 0 & 2 & 0 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix} = -2\begin{vmatrix} 2 & 4 \\ 4 & 6 \end{vmatrix} = -2\ (2 \times 6 - 4 \times 4)$$

$$= -2\ (12 - 16) = (-2) \times (-4) = 8.$$

Example 6. *Evaluate* $\begin{vmatrix} 3 & 7 & 13 \\ -5 & 0 & 0 \\ 0 & 11 & -2 \end{vmatrix}$.

Solution. As the second row contains two zeros, expanding the given determinant by 2nd row, we get

$$\begin{vmatrix} 3 & 7 & 13 \\ -5 & 0 & 0 \\ 0 & 11 & -2 \end{vmatrix} = -(-5)\begin{vmatrix} 7 & 13 \\ 11 & -2 \end{vmatrix} + 0\begin{vmatrix} 3 & 13 \\ 0 & -2 \end{vmatrix} - 0\begin{vmatrix} 3 & 7 \\ 0 & 11 \end{vmatrix}$$

$$= 5\ (-14 - 143) + 0 - 0 = -785.$$

Example 7. *Evaluate the following determinants :*

(*i*) $\begin{vmatrix} 2 & 3 & -5 \\ 7 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix}$ (*ii*) $\begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$ (*NCERT*)

Solution. (*i*) $\begin{vmatrix} 2 & 3 & -5 \\ 7 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 3\begin{vmatrix} 7 & -2 \\ -3 & 1 \end{vmatrix} + (-5)\begin{vmatrix} 7 & 1 \\ -3 & 4 \end{vmatrix}$ (Expanding by R_1)

$$= 2\,(1 - (-8)) - 3\,(7 - 6) - 5\,(28 - (-3))$$

$$= 2.9 - 3.1 - 5.31 = 18 - 3 - 155 = -140.$$

(*ii*) $\begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$

$$= 0 \times \begin{vmatrix} 0 & \sin\beta \\ -\sin\beta & 0 \end{vmatrix} - \sin\alpha\begin{vmatrix} -\sin\alpha & \sin\beta \\ \cos\alpha & 0 \end{vmatrix} + (-\cos\alpha)\begin{vmatrix} -\sin\alpha & 0 \\ \cos\alpha & -\sin\beta \end{vmatrix}$$

(Expanding by R_1)

$$= 0 - \sin\alpha\ (0 - \cos\alpha \sin\beta) - \cos\alpha\ (\sin\alpha \sin\beta - 0)$$

$$= \sin\alpha \cos\alpha \sin\beta - \sin\alpha \cos\alpha \sin\beta$$

$$= 0.$$

Example 8. *There are two values of x which make determinant* $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & x & -1 \\ 0 & 4 & 2x \end{vmatrix} = 86$, *find the sum of these numbers.* *(NCERT Examplar Problems)*

Solution. Given $\begin{vmatrix} 1 & -2 & 5 \\ 2 & x & -1 \\ 0 & 4 & 2x \end{vmatrix} = 86$ (Expanding by C_1)

$$\Rightarrow \quad 1\begin{vmatrix} x & -1 \\ 4 & 2x \end{vmatrix} - 2\begin{vmatrix} -2 & 5 \\ 4 & 2x \end{vmatrix} + 0\begin{vmatrix} -2 & 5 \\ x & -1 \end{vmatrix} = 86$$

$\Rightarrow \quad 1(2x^2 + 4) - 2(-4x - 20) = 86 \Rightarrow 2x^2 + 8x - 42 = 0$

$\Rightarrow \quad x^2 + 4x - 21 = 0.$

Let α, β be the roots of this equation, then $\alpha + \beta = -\frac{4}{1} = -4$.

Hence, the sum of the two values of $x = -4$.

Example 9. *Show that the value of the determinant* $\begin{vmatrix} a & \sin x & \cos x \\ -\sin x & -a & 1 \\ \cos x & 1 & a \end{vmatrix}$ *is independent of x.* *(NCERT)*

Solution. Expanding the given determinant by the first row, we get

$$\begin{vmatrix} a & \sin x & \cos x \\ -\sin x & -a & 1 \\ \cos x & 1 & a \end{vmatrix} = a\begin{vmatrix} -a & 1 \\ 1 & a \end{vmatrix} - \sin x\begin{vmatrix} -\sin x & 1 \\ \cos x & a \end{vmatrix} + \cos x\begin{vmatrix} -\sin x & -a \\ \cos x & 1 \end{vmatrix}$$

$= a(-a^2 - 1) - \sin x(-a \sin x - \cos x) + \cos x(-\sin x + a \cos x)$

$= -a^3 - a + a \sin^2 x + \sin x \cos x - \sin x \cos x + a \cos^2 x$

$= -a^3 - a + a(\sin^2 x + \cos^2 x) = -a^3 - a + a \times 1 = -a^3$,

which is independent of x.

Example 10. *Find the minors and cofactors of each element of the second column of the determinant Δ and hence find the value of the determinant Δ where*

$$\Delta = \begin{vmatrix} 3 & -2 & 1 \\ 4 & 6 & 5 \\ 2 & -1 & 7 \end{vmatrix}.$$

Solution. $M_{12} = \begin{vmatrix} 4 & 5 \\ 2 & 7 \end{vmatrix} = 28 - 10 = 18$, $\quad M_{22} = \begin{vmatrix} 3 & 1 \\ 2 & 7 \end{vmatrix} = 21 - 2 = 19$ and

$M_{32} = \begin{vmatrix} 3 & 1 \\ 4 & 5 \end{vmatrix} = 15 - 4 = 11.$

$\therefore \quad A_{12} = (-1)^{1+2} M_{12} = (-1).18 = -18,$

$A_{22} = (-1)^{2+2} M_{22} = 1.19 = 19$ and

$A_{32} = (-1)^{3+2} M_{32} = (-1).11 = -11.$

Now, expanding the given determinant by 2nd column, we get

$\Delta = (-2).(-18) + 6.19 + (-1).(-11)$

$= 36 + 114 + 11 = 161.$

Example 11. *Find the cofactors of the elements of the third row of the determinant* $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ *and verify that* $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$. (*NCERT*)

Solution. $M_{31} = \begin{vmatrix} -3 & 5 \\ 0 & 4 \end{vmatrix} = -12 - 0 = -12$, $M_{32} = \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = 8 - 30 = -22$,

$M_{33} = \begin{vmatrix} 2 & -3 \\ 6 & 0 \end{vmatrix} = 0 - (-18) = 18.$

$\therefore$ $A_{31} = (-1)^{3+1} M_{31} = -12$, $A_{32} = (-1)^{3+2} M_{32} = 22$ and $A_{33} = (-1)^{3+3} M_{33} = 18$.

Here $a_{11} = 2$, $a_{12} = -3$ and $a_{13} = 5$.

$\therefore$ $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 2(-12) + (-3)(22) + 5(18)$

$= -24 - 66 + 90 = 0.$

EXERCISE 4.1

Very short answer type questions (1 to 12) :

1. Evaluate the following determinants :

(*i*) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$ (*NCERT*) (*ii*) $\begin{vmatrix} \cos 90° & -\cos 45° \\ \sin 90° & \sin 45° \end{vmatrix}$ (*iii*) $\begin{vmatrix} 2\cos\theta & -2\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$.

2. Evaluate the following determinants :

(*i*) $\begin{vmatrix} \sin 30° & \cos 30° \\ -\sin 60° & \cos 60° \end{vmatrix}$ (*C.B.S.E. 2008*) (*ii*) $\begin{vmatrix} -1 & 31 & 40 \\ 0 & 5 & -432 \\ 0 & 0 & 20 \end{vmatrix}$

3. (*i*) Find the value of $\begin{vmatrix} \sin A & -\sin B \\ \cos A & \cos B \end{vmatrix}$, where A = 63° and B = 27°.

(*ii*) Find the value of $\begin{vmatrix} \cos A & \sin A \\ -\sin B & \cos B \end{vmatrix}$, where A = 75° and B = 45°.

4. (*i*) Write the cofactor of a_{12} in the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$. (*C.B.S.E. 2012, 08*)

(*ii*) If A_{ij} is the cofactor of the element a_{ij} of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$, then write the value of $a_{32} A_{32}$. (*C.B.S.E. 2013*)

5. If $A = [a_{ij}]$ is a 3×3 matrix and A_{ij}'s denote the cofactors of the corresponding elements a_{ij}'s, then write the expression for the value of $|A|$ by expanding $|A|$ by

(*i*) 2nd row (*ii*) third column.

6. Find the value of x if

(*i*) $\begin{vmatrix} 2x+5 & 3 \\ 5x+2 & 9 \end{vmatrix} = 0$ (*C.B.S.E. 2008*) (*ii*) $\begin{vmatrix} x & 4 \\ 2 & 2x \end{vmatrix}$ (*C.B.S.E. 2009*) (*iii*) $\begin{vmatrix} x+2 & 3 \\ x+5 & 4 \end{vmatrix} = 3$. (*C.B.S.E. 2008*)

7. Find the value of x if

(*i*) $\begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix}$ (*NCERT*) (*ii*) $\begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$. (*NCERT*)

8. (*i*) If $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$, then write the value of x. (*C.B.S.E. 2013*)

(*ii*) If $\begin{vmatrix} \sqrt{6} & x \\ \sqrt{20} & \sqrt{24} \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 5 & 2 \end{vmatrix}$, then write the value of x. (*C.B.S.E. 2013*)

9. (*i*) If $x \in$ **N** and $\begin{vmatrix} x & 3 \\ 4 & x \end{vmatrix} = \begin{vmatrix} 4 & -3 \\ 0 & 1 \end{vmatrix}$, find the value(s) of x.

(*ii*) If $\begin{vmatrix} x & x \\ 1 & x \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$, write the positive value of x. (*C.B.S.E. 2011*)

10. If $x \in$ **I** and $\begin{vmatrix} 2x & 3 \\ -1 & x \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ x & 3 \end{vmatrix}$, find the value(s) of x.

11. If $x \in$ **R**, $0 \le x \le \frac{\pi}{2}$, and $\begin{vmatrix} 2\sin x & -1 \\ 1 & \sin x \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ -4 & \sin x \end{vmatrix}$, then find the values of x.

12. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that $|2A| = 4|A|$. (*NCERT*)

13. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ -1 & 7 \end{bmatrix}$, find the determinant of the matrix $3A^2 - 2B$.

14. Evaluate the following determinants :

(*i*) $\begin{vmatrix} 2 & 4 & 1 \\ 8 & 5 & 2 \\ -1 & 3 & 7 \end{vmatrix}$ (*ii*) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$ (*iii*) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$.

(*NCERT*) (*NCERT*)

15. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, then show that $|3A| = 27|A|$. (*NCERT*)

16. Find the integral value(s) of x if $\begin{vmatrix} x^2 & x & 1 \\ 0 & 2 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 28$.

17. Prove that $\begin{vmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$.

18. Show that the value of the determinant $\begin{vmatrix} 0 & \tan x & 1 \\ 1 & -\sec x & 0 \\ \sec x & 0 & \tan x \end{vmatrix}$ is independent of x.

19. Using cofactors of elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$. (*NCERT*)

4.2 PROPERTIES OF DETERMINANTS

The properties of determinants serve the purpose of useful tools for computing the values of the given determinants. The proofs of most of the properties of determinants are beyond the scope of the present book. Therefore, here we shall state these properties and verify them by taking some examples.

Property 1. *If each element in a row or in a column of a determinant is zero, then the value of the determinant is zero.*

Verification. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix}$ be a determinant in which each element in the second row is zero.

Expanding Δ by the second row, we get

$$\Delta = -0\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + 0\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - 0\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = 0.$$

Property 2. *If each element on one side of the principal diagonal of a determinant is zero, then the value of the determinant is the product of the diagonal elements.*

Verification. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{vmatrix}$ be a determinant in which all elements on one side of the principal diagonal are zero.

Expanding Δ by C_1, we get

$$\Delta = a_1\begin{vmatrix} b_2 & c_2 \\ 0 & c_3 \end{vmatrix} - 0\begin{vmatrix} b_1 & c_1 \\ 0 & c_3 \end{vmatrix} + 0\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - 0.c_2) - 0 + 0 = a_1 b_2 c_3 .$$

Property 3. *The value of a determinant remains unchanged if its rows and columns are interchanged.*

Verification. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and Δ_1 be the determinant obtained from Δ by interchanging its rows and columns

i.e. $\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Expanding Δ by R_1, we get

$$\Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 \quad \ldots(i)$$

Expanding Δ_1 by R_1, we get

$$\Delta_1 = a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 \quad \ldots(ii)$$

From (*i*) and (*ii*), we get $\Delta = \Delta_1$.

REMARK

It follows from the above property that if A is a square matrix, then det A′ = det A *i.e.* $|A'| = |A|$ where A′ is the transpose of A.

Property 4. *If any two rows (or columns) of a determinant are interchanged, then the value of the determinant changes by minus sign only.*

Verification. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and Δ_1 be the determinant obtained from Δ by interchanging its first and third columns

i.e. $\Delta_1 = \begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}$

Expanding Δ by R_1, we get

$$\Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$
$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 \quad \ldots(i)$$

Expanding Δ_1 by R_1, we get

$$\Delta_1 = c_1(b_2a_3 - b_3a_2) - b_1(c_2a_3 - c_3a_2) + a_1(c_2b_3 - c_3b_2)$$
$$= a_3b_2c_1 - a_2b_3c_1 - a_3b_1c_2 + a_2b_1c_3 + a_1b_3c_2 - a_1b_2c_3$$
$$= -(a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1) \quad \ldots(ii)$$

From (*i*) and (*ii*), we get $\Delta_1 = -\Delta$.

Corollary. *If any row (or column) of a determinant Δ be passed over m rows (or columns), then the resulting determinant $\Delta_1 = (-1)^m \Delta$.*

Verification. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and Δ_1 be the determinant obtained from Δ by passing over its first column over the next two columns

i.e. $$\Delta_1 = \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix}$$

Let us interchange 1st and 3rd columns in Δ_1, then by property 4 we get

$$\Delta_1 = -\begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} = (-1)\begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix}.$$

Now, on interchanging 2nd and 3rd columns, we get

$$\Delta_1 = (-1)^2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{(Using property 4 again)}$$

$$\Rightarrow \quad \Delta_1 = (-1)^2 \Delta.$$

Property 5. *If two parallel lines (rows or columns) of a determinant are identical (all corresponding elements are same), then the value of the determinant is zero.*

Verification. Let Δ be the given determinant which has two parallel lines identical, say first and third rows, then

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}$$

Expanding Δ by R_1, we get

$$\Delta = a_1(b_2c_1 - b_1c_2) - b_1(a_2c_1 - a_1c_2) + c_1(a_2b_1 - a_1b_2)$$
$$= a_1b_2c_1 - a_1b_1c_2 - a_2b_1c_1 + a_1b_1c_2 + a_2b_1c_1 - a_1b_2c_1$$
$$= 0.$$

Proof. If we interchange the identical parallel lines (rows or columns) of the determinant Δ, then Δ does not change.

But by property 4 it follows that Δ has changed its sign, therefore,

$$\Delta = -\Delta$$

$$\Rightarrow \quad 2\Delta = 0 \Rightarrow \Delta = 0.$$

Property 6. *If each element of a row (or a column) of a determinant is multiplied by the same number k, then the value of the new determinant is k times the value of the original determinant.*

Verification. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and Δ_1 be the determinant obtained from Δ by multiplying every element of second row by the same number k *i.e.*

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_2 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Expanding Δ by R_1, we get

$$\Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \quad \ldots(i)$$

Expanding Δ_1 by R_1, we get

$$\Delta_1 = a_1(kb_2c_3 - kb_3c_2) - b_1(ka_2c_3 - ka_3c_2) + c_1(ka_2b_3 - ka_3b_2)$$

$$= k[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)] \quad \ldots(ii)$$

From (*i*) and (*ii*), we get $\Delta_1 = k\,\Delta$.

REMARKS

1. By the above property, we can take out any common factor from any one row or any one column of a given determinant.
2. We know that kA is matrix obtained by multiplying each element of matrix A with number k, therefore, if A is a square matrix of order n, then $|kA| = k^n |A|$ (Because the common factor k will be taken out from each of n-rows or n-columns).

   Thus, if each element of a determinant Δ is multiplied by the same number k and Δ_1 is the new determinant then

   $\Delta_1 = k\,\Delta$ if order of $\Delta = 1$

   $\Delta_1 = k^2\,\Delta$ if order of $\Delta = 2$

   $\Delta_1 = k^3\,\Delta$ if order of $\Delta = 3$ etc.
3. If A is a skew-symmetric matrix of order n, then $A' = -A$

   $\Rightarrow \quad |A'| = |-A| = (-1)^n\ |A|$ (By the above remark)

   In particular, if n is odd, then

   $|A'| = -|A|$ $\quad (\because (-1)^{\text{odd integer}} = -1)$

   But $\quad |A'| = |A|$ (By property 3)

   $\Rightarrow \quad |A| = -|A| \Rightarrow 2|A| = 0$

   $\Rightarrow \quad |A| = 0.$

   Hence, the determinant of a skew-symmetric matrix of odd order is zero.

Corollary. *If two parallel lines (rows or columns) of a determinant are such that the elements of one line are equi-multiples of the elements of the other line, then the value of the determinant is zero.* (Use properties 6 and 5)

Property 7. *If each element of a row (or a column) of a determinant consists of sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants whose other rows (or columns) are not altered.*

Verification. Let $\Delta = \begin{vmatrix} a_1 + d_1 & b_1 & c_1 \\ a_2 + d_2 & b_2 & c_2 \\ a_3 + d_3 & b_3 & c_3 \end{vmatrix}$. Here, each element in the first column consists of the sum of two terms.

Expanding Δ by C_1, we get

$$\begin{aligned}\Delta &= (a_1 + d_1)(b_2c_3 - b_3c_2) - (a_2 + d_2)(b_1c_3 - b_3c_1) + (a_3 + d_3)(b_1c_2 - b_2c_1)\\ &= [a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)]\\ &\quad + [d_1(b_2c_3 - b_3c_2) - d_2(b_1c_3 - b_3c_1) + d_3(b_1c_2 - b_2c_1)]\\ &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}.\end{aligned}$$

Property 8. *If to each element of a row (or a column) of a determinant be added the equi-multiples of the corresponding elements of one or more rows (or columns), the value of the determinant remains unchanged.*

Verification. Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ and Δ_1 be the determinant obtained from Δ by adding k times the elements of second column to the corresponding elements of the first column *i.e.*

$$\Delta_1 = \begin{vmatrix} a_1 + k\,a_2 & a_2 & a_3 \\ b_1 + k\,b_2 & b_2 & b_3 \\ c_1 + k\,c_2 & c_2 & c_3 \end{vmatrix}$$

By using property 7, we get

$$\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} k\,a_2 & a_2 & a_3 \\ k\,b_2 & b_2 & b_3 \\ k\,c_2 & c_2 & c_3 \end{vmatrix}$$

$$= \Delta + k \begin{vmatrix} a_2 & a_2 & a_3 \\ b_2 & b_2 & b_3 \\ c_2 & c_2 & c_3 \end{vmatrix} \qquad \text{(Using property 6)}$$

$$= \Delta + k \,.\, 0 \qquad \text{(Using property 5)}$$

$$= \Delta.$$

Property 9. *The sum of the products of elements of any row (or column) with the cofactors of the corresponding elements of some other row (or column) is zero.*

Verification. Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

Then the sum of the products of elements of first row with the cofactors of the corresponding elements of the third row

$$\begin{aligned}&= a_{11}\,A_{31} + a_{12}\,A_{32} + a_{13}\,A_{33}\\ &= a_{11}\,(-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a_{12}(-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{13}(-1)^{3+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\\ &= a_{11}\,(a_{12}\,a_{23} - a_{22}\,a_{13}) - a_{12}\,(a_{11}\,a_{23} - a_{21}\,a_{13}) + a_{13}\,(a_{11}\,a_{22} - a_{21}\,a_{12})\\ &= 0.\end{aligned}$$

Property 10. *If A and B are square matrices of same order, then det AB = det A . det B.*

Verification. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$,

then $$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{bmatrix}$$

$$\Rightarrow \quad \det AB = \begin{vmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{vmatrix}$$

$$= (a\alpha + b\gamma)(c\beta + d\delta) - (c\alpha + d\gamma)(a\beta + b\delta)$$

$$= ac\alpha\beta + ad\alpha\delta + bc\beta\gamma + bd\gamma\delta - ac\alpha\beta - bc\alpha\delta - ad\beta\gamma - bd\gamma\delta$$

$$= ad\alpha\delta - bc\alpha\delta + bc\beta\gamma - ad\beta\gamma$$

$$= (ad - bc)\alpha\delta - (ad - bc)\beta\gamma$$

$$= (ad - bc)(\alpha\delta - \beta\gamma)$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \det A. \det B.$$

4.2.1 Elementary operations

Let Δ be a determinant of order n, $n \geq 2$; R_1, R_2, R_3, ... denote its first row, second row, third row, ... and C_1, C_2, C_3, ... denote its first column, second column, third column, ... respectively.

(*i*) The operation of interchanging the ith row and jth row of Δ will be denoted by $R_i \leftrightarrow R_j$ and the operation of interchanging the ith column and jth column of Δ will be denoted by $C_i \leftrightarrow C_j$.

(*ii*) The operation of multiplying each element of the ith row of Δ by a number k will be denoted by $R_i \to k R_i$ and the operation of multiplying each element of the ith column of Δ by a number k will be denoted by $C_i \to k C_i$.

(*iii*) The operation of adding to each element of the ith row of Δ, k times the corresponding elements of the jth row ($j \neq i$) will be denoted by $R_i \to R_i + k R_j$ and the operation of adding to each element of the ith column of Δ, k times the corresponding elements of the jth column ($j \neq i$) will be denoted by $C_i \to C_i + k C_j$.

NOTE

If we apply $R_i \to k R_i$ or $C_i \to k C_i$, then multiply the determinant by $\frac{1}{k}$ (in the same step).

ILLUSTRATIVE EXAMPLES

Example 1. *Without expanding, evaluate the following determinants :*

(*i*) $\begin{vmatrix} 49 & 1 & 6 \\ 39 & 7 & 4 \\ 26 & 2 & 3 \end{vmatrix}$ (*ii*) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ (*iii*) $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$

(*C.B.S.E. 2009*)

Solution. (*i*) Operating $C_1 \to C_1 - 8C_3$ (property 8), we get

$$\begin{vmatrix} 49 & 1 & 6 \\ 39 & 7 & 4 \\ 26 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 49 - 8.6 & 1 & 6 \\ 39 - 8.4 & 7 & 4 \\ 26 - 8.3 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 6 \\ 7 & 7 & 4 \\ 2 & 2 & 3 \end{vmatrix} = 0 \qquad \text{(By property 5)}$$

(*ii*) Operating $C_3 \to C_3 + C_2$ (property 8), we get

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix}$$

ANSWERS

EXERCISE 4.1

1. (*i*) $x^3 - x^2 + 2$ (*ii*) $\frac{1}{\sqrt{2}}$ (*iii*) 2 **2.** (*i*) 1 (*ii*) –100

3. (*i*) 1 (*ii*) $\frac{\sqrt{3}}{2}$ **4.** (*i*) 46 (*ii*) 110

5. (*i*) $a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$ (*ii*) $a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$

6. (*i*) –13 (*ii*) 2, –2 (*iii*) 10 **7.** (*i*) $\pm\sqrt{3}$ (*ii*) 2

8. (*i*) 2 (*ii*) $\sqrt{5}$ **9.** (*i*) 4 (*ii*) 2 **10.** –2

11. $\frac{\pi}{6}, \frac{\pi}{2}$ **13.** 727 **14.** (*i*) –145 (*ii*) 46 (*iii*) 5

16. 2 **19.** 7

EXERCISE 4.2

2. 0 **3.** (*i*) 0 (*ii*) 0 (*iii*) 0 (*iv*) 0

4. (*i*) 0 (*ii*) 0 (*iii*) 0 (*iv*) 0

5. (*i*) 0 (*ii*) 0 (*iii*) 0 (*iv*) 0

6. (*i*) 0 (*ii*) 0 (*iii*) 0 (*iv*) 0

7. (*i*) –45 (*ii*) 32 (*iii*) 135 (*iv*) 250 (*v*) 108

8. –126 **9.** (*i*) ±4 (*ii*) ±3

10. (*i*) $a_1b_2 - a_2b_1$ (*ii*) $a_1b_2 - a_2b_1$ (*iii*) 0 (*iv*) 0

11. (*i*) Value of |A| (*ii*) Value of |A| (*iii*) 0 (*iv*) 0

12. $-\frac{a}{3}$ **15.** (*i*) –676 (*ii*) – 8

17. (*i*) 0, 0, – $(a + b + c)$ (*ii*) 2, 3, 6 **18.** (*ii*) 0 (*iv*) 0

42. (*i*) $\frac{2}{3}, \frac{11}{3}, \frac{11}{3}$ (*ii*) 1, 1, –9 (*iii*) 2, 1, –3 (*iv*) 4

EXERCISE 4.3

1. (*i*) 30.5 sq. units (*ii*) 23.5 sq. units

3. Do not form a triangle **4.** –5 **5.** $x + y - 2 = 0$

6. 3, –3 **7.** 0, 8 **8.** –2, 12 **12.** $y = 2x$

EXERCISE 4.4

1. (*i*) 3 (*ii*) 1 (*iii*) $\frac{\pi}{3}, \frac{2\pi}{3}$ (*iv*) $\frac{\pi}{6}, \frac{5\pi}{6}$

2. Matrix in (*ii*) **3.** (*i*) $\begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

4. (*i*) $\frac{10}{3}$ (*ii*) $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix}$ (*iv*) $\frac{1}{7}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

5. (*i*) 25 (*ii*) 8, –8 **6.** (*i*) 7 (*ii*) 576 (*iii*) 25

7. (*i*) 9 (*ii*) a^9 (*iii*) $\begin{bmatrix} 3 & -1 \\ 4 & 5 \end{bmatrix}$

10. (*i*) $\begin{bmatrix} yz & 0 & 0 \\ 0 & zx & 0 \\ 0 & 0 & xy \end{bmatrix}$ (*ii*) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$ **11.** $\begin{bmatrix} 26 & 0 \\ 0 & 26 \end{bmatrix}$