APPLICATIONS OF INTEGRALS

8

INTRODUCTION

In geometry, we learnt some formulae to calculate areas of various geometrical figures such as triangles, quadrilaterals, polygons and circles etc., which are very useful in many real life problems. However, by these formulae of geometry, we can calculate areas of only simple figures. They are inadequate for calculating the areas enclosed by closed curves. For that we need definite integrals. In fact, definite integrals have a wide range of applications.

8.1 AREAS OF BOUNDED REGIONS

If the function f is continuous and non-negative in the closed interval $[a, b]$, then the area of the region below the curve $y = f(x)$, above the x-axis and between the ordinates $x = a$ and $x = b$ or briefly the area of the region bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is given by $\int_a^b f(x)\,dx$ or $\int_a^b y\,dx$.

Proof. Let AB be the curve $y = f(x)$ between AC($x = a$) and BD($x = b$), then the required area is the area of the shaded region ACDB.

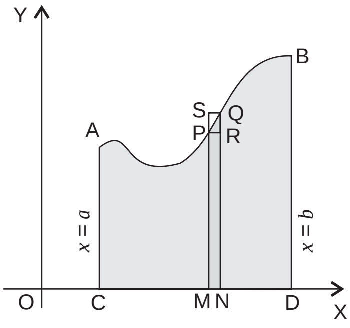


Fig. 8.1.

Let P(x, y) be a point on the curve $y = f(x)$ and Q($x + \delta x$, $y + \delta y$) be a neighbouring point on the curve, then MP = y, NQ = $y + \delta y$ and MN = δx. Let A be the area of the region ACMP and A + δA be the area of the region ACNQ, then δA = area of region PMNQ.

Area of rectangle PMNR = $y\delta x$ and area of rectangle SMNQ = $(y + \delta y)\delta x$.

From fig. 8.1, area of rectangle PMNR $\le$ area of region PMNQ $\le$ area of rectangle SMNQ

$$\Rightarrow \quad y\delta x \le \delta A \le (y + \delta y)\delta x$$

$$\Rightarrow \quad y \le \frac{\delta A}{\delta x} \le y + \delta y \qquad \ldots(i)$$

When P $\to$ Q, $\delta x \to 0$, $\delta y \to 0$ and $\frac{\delta A}{\delta x} \to \frac{dA}{dx}$.

From (i), $\underset{\delta x \to 0}{\text{Lt}}\, y \le \underset{\delta x \to 0}{\text{Lt}} \frac{\delta A}{\delta x} \le \underset{\delta y \to 0}{\text{Lt}} (y + \delta y)$

$$\Rightarrow \quad y \le \frac{dA}{dx} \le y \Rightarrow y = \frac{dA}{dx}.$$

Integrating both sides w.r.t. x between the limits a to b, we get

$$\int_a^b y\,dx = \int_a^b \frac{dA}{dx}\,dx = [A]_a^b$$

$$= \text{(value of area A when } x = b) - \text{(value of area A when } x = a)$$

$$= \text{area ACDB} - 0 = \text{area ACDB.}$$

If a function f is continuous and non-positive in the closed interval $[a, b]$, then the curve $y = f(x)$ lies below the x-axis and the definite integral $\int_a^b f(x)\,dx$ is negative. Since the area of a region is always non-negative, the area of the region bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is given by $\left|\int_a^b f(x)\,dx\right|$ or $\left|\int_a^b y\,dx\right|$.

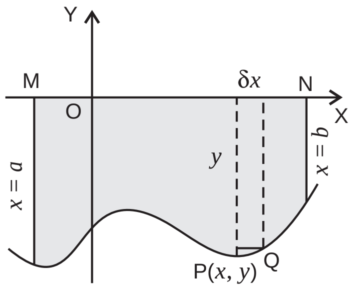


Fig. 8.2.

Hence, if the curve $y = f(x)$ is continuous and does not cross the x-axis, then the area of the region bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ is given by $\left|\int_a^b f(x)\,dx\right|$ or $\left|\int_a^b y\,dx\right|$.

Similarly, if the curve $x = g(y)$ is continuous and does not cross the y-axis, then the area of the region bounded by the curve $x = g(y)$, the y-axis and the abscissae $y = c$, $y = d$ is given by

$$\left|\int_c^d g(y)\,dy\right| \text{ or } \left|\int_c^d x\,dy\right|.$$

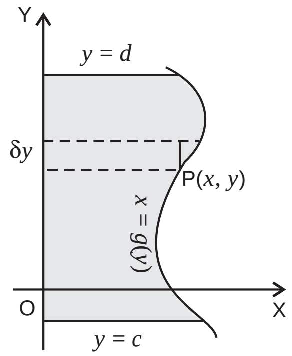


Fig. 8.3.

REMARK

It may be noted that when sign of $f(x)$ is not known, then $\int_a^b f(x)\,dx$ may not represent the area *enclosed between* the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$, whereas $\int_a^b |f(x)|\,dx$ equals the area enclosed between the graph of the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$.

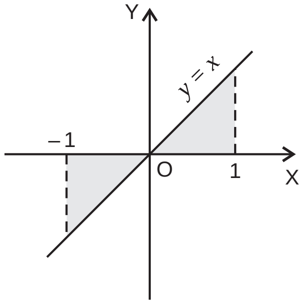
For example, let us consider the integrals $\int_{-1}^{1} x\,dx$ and $\int_{-1}^{1} |x|\,dx$.

First integral $= \int_{-1}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{2}(1^2 - (-1)^2) = 0$, whereas second integral

$$= \int_{-1}^{1} |x|\,dx = \int_{-1}^{0} (-x)\,dx + \int_{0}^{1} x\,dx$$

(Common sense suggests this division as $|x| = -x$ in $[-1, 0]$ and $|x| = x$ in $[0, 1]$).

$$= \left[-\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1} = -\frac{1}{2}(0-1) + \frac{1}{2}(1-0) = 1.$$



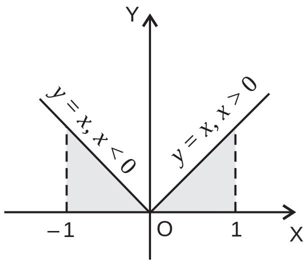


Fig. 8.4.

Clearly, the area enclosed between $y = x$, the x-axis and the ordinates $x = -1$ and $x = 1$ is not zero.

It follows that if the graph of a function f is continuous in [a, b] and crosses the x-axis at finitely many points in [a, b], then the area enclosed between the graph of the curve y = f (x), the x-axis and the ordinates x = a, x = b is given by $\int_a^b |f(x)|\,dx$ ***or*** $\int_a^b |y|\,dx$.

8.2 AREA BETWEEN TWO CURVES

If $f(x)$, $g(x)$ are both continuous in $[a, b]$ and $0 \le g(x) \le f(x)$ for all $x \in [a, b]$, then the area of the region between the graphs of $y = f(x)$, $y = g(x)$ and the ordinates $x = a$, $x = b$ is given by

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b (f(x) - g(x))\,dx.$$

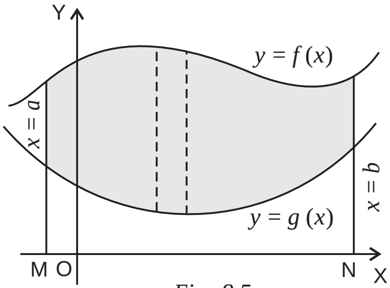


Fig. 8.5.

Similarly, the area of the region between the graphs of $x = f(y)$, $x = g(y)$ and the abscissae $y = c$, $y = d$ is given by

$$\int_c^d (f(y) - g(y))\,dy.$$

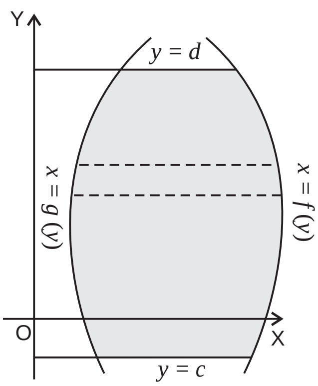


Fig. 8.6.

REMARKS

1. If $f(x)$, $g(x)$ are both continuous in $[a, b]$ and $g(x) \le f(x)$ for all $x \in [a, b]$, then the above formula also holds when one or both of the curves $y = f(x)$ and $y = g(x)$ lie partially or completely below the x-axis.

2. If the graphs of the curves $y = f(x)$ and $y = g(x)$ cross each other at finitely many points, then the area enclosed between the graphs of the two curves and the ordinates $x = a$ and $x = b$ is given by $\int_a^b |f(x) - g(x)|\, dx$.

3. Similarly, the area of the region between the graphs of $x = f(y)$, $x = g(y)$ and the abscissae $y = c$, $y = d$ is given by $\int_c^b |f(y) - g(y)|\, dy$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the area of the region bounded by $y^2 = 4x$, $x = 1$, $x = 4$ and the x-axis in the first quadrant.* *(C.B.S.E. 2006 C, 06, 04)*

Solution. The given curve is $y^2 = 4x$ which represents a right hand parabola with vertex at (0, 0). The area bounded by $y^2 = 4x$, $x = 1$, $x = 4$ and the x-axis is shown shaded in fig. 8.7.

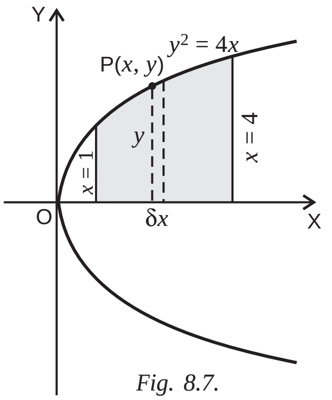


Fig. 8.7.

Required area $= \int_1^4 y\, dx = \int_1^4 2\sqrt{x}\, dx$

$(\because\ y^2 = 4x \Rightarrow y = 2\sqrt{x}$ in the first quadrant$)$

$$= 2 \cdot \left[\frac{x^{3/2}}{\frac{3}{2}}\right]_1^4 = \frac{4}{3}\left[4^{3/2} - 1^{3/2}\right]$$

$$= \frac{4}{3}\,[8 - 1] = \frac{28}{3} \text{ sq. units.}$$

Example 2. *Determine the area enclosed between the curve $y = 4x - x^2$ and the x-axis.*

Solution. Given curve is $y = 4x - x^2$.

It can be written as $x^2 - 4x = -y \Rightarrow (x - 2)^2 = -(y - 4)$

which represents a downward parabola with vertex at (2, 4).

The parabola meets x-axis *i.e.* $y = 0$ at $4x - x^2 = 0$ *i.e.* at $x = 0$, $x = 4$. The area enclosed between the curve and the x-axis is shown shaded in fig. 8.8.

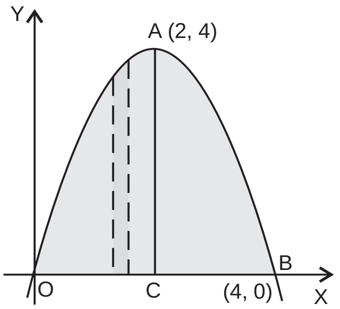


Fig. 8.8.

$\therefore$ Required area $= \int_0^4 y\, dx = \int_0^4 (4x - x^2)\, dx = \left[4 \cdot \frac{x^2}{2} - \frac{x^3}{3}\right]_0^4$

$$= \left(32 - \frac{64}{3}\right) - (0 - 0) = \frac{32}{3} \text{ sq. units.}$$

Alternatively. Since the parabola is symmetrical about the line $x = 2$,

$$\text{required area} = 2\int_0^2 (4x - x^2)\, dx = 2\left[4 \cdot \frac{x^2}{2} - \frac{x^3}{3}\right]_0^2$$

$$= 2\left[\left(8 - \frac{8}{3}\right) - (0 - 0)\right] = 2 \cdot \frac{16}{3} = \frac{32}{3} \text{ sq. units.}$$

REMARK

In case of symmetrical closed area, find the area of the smallest part and multiply the result by the number of symmetrical parts.

Example 3. *Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.* (*NCERT*)

Solution. The given curve is $y^2 = 4x$ which represents a right hand parabola with vertex (0, 0). The area bounded by the curve $y^2 = 4x$ and the line $x = 3$ is shown shaded in fig. 8.9. Since the given area is symmetrical about x-axis,

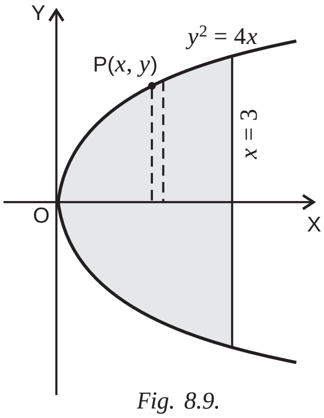


Fig. 8.9.

required area = 2 (area of the region bounded by $y^2 = 4x$, the axis and the line $x = 3$)

$$= 2\int_0^3 y\,dx = 2\int_0^3 2\sqrt{x}\,dx$$

($\because$ $y^2 = 4x \Rightarrow y = 2\sqrt{x}$ in the first quadrant)

$$= 4\,.\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^3 = \frac{8}{3}\left[3^{3/2} - 0\right] = \frac{8}{3}\,.\,3\sqrt{3} = 8\sqrt{3} \text{ sq. units.}$$

Example 4. *Find the area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$.* (*NCERT*)

Solution. The given curve is $y^2 = 4x$ which represents a right hand parabola with vertex (0, 0). The area bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is shown shaded in fig. 8.10.

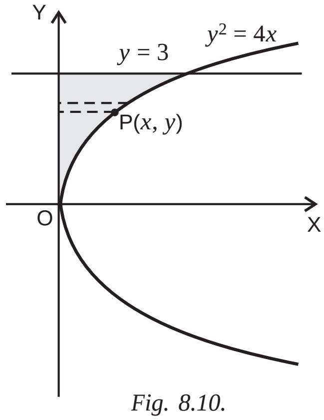


Fig. 8.10.

$$\text{Required area} = \int_0^3 x\,dy = \int_0^3 \frac{y^2}{4}\,dy$$

($\because$ $y^2 = 4x \Rightarrow x = \frac{y^2}{4}$)

$$= \frac{1}{4}\,.\left[\frac{y^3}{3}\right]_0^3 = \frac{1}{12}\,[27 - 0]$$

$$= \frac{9}{4} \text{ sq. units.}$$

Example 5. *Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$.* (*NCERT*)

Solution. The given curve is $y = x^2$ which represents an upward parabola with vertex at (0, 0). The area bounded by the curve and the line $y = 4$ is shown shaded in fig. 8.11.

Since the area is symmetrical about y-axis,

required area = 2 (area of the region bounded by $y = x^2$, the y-axis and the line $y = 4$)

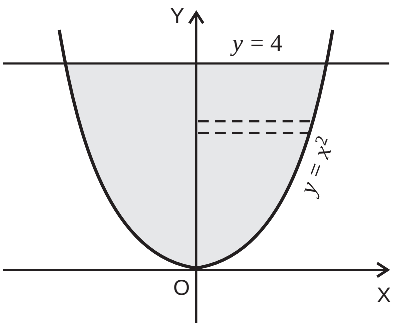


Fig. 8.11.

$$= 2\int_0^4 x\,dy = 2\int_0^4 \sqrt{y}\,dx$$

($\because$ $x^2 = y \Rightarrow x = \sqrt{y}$ in the first quadrant)

$$= 2\,.\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^4 = \frac{4}{3}\left[4^{3/2} - 0\right] = \frac{4}{3}\,[8 - 0] = \frac{32}{3} \text{ sq. units.}$$

Example 6. *Find the area of the region bounded by the curve $y = x^2 - 6x + 5$, the x-axis and the lines $x = 2$, $x = 4$.*

Solution. The given curve is

$y = x^2 - 6x + 5$. It can be written as

$y = (x^2 - 6x + 9) - 4$

$\Rightarrow$ $y - (-4) = (x - 3)^2$, which represents an upward parabola with vertex at (3, –4). The required area is shown shaded in fig. 8.12.

As the required portion of the curve lies below x-axis, y is negative, therefore, required area

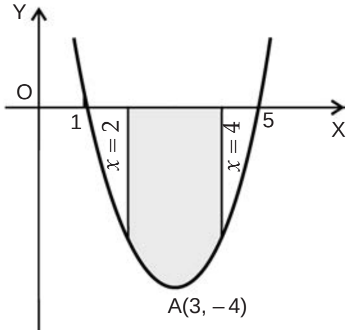


Fig. 8.12.

$$= \left|\int_2^4 y\,dx\right| = \left|\int_2^4 (x^2 - 6x + 5)\,dx\right| = \left|\left[\frac{x^3}{3} - 6.\frac{x^2}{2} + 5x\right]_2^4\right|$$

$$= \left|\left(\frac{64}{3} - 48 + 20\right) - \left(\frac{8}{3} - 12 + 10\right)\right| = \left|-\frac{20}{3} - \frac{2}{3}\right| = \frac{22}{3} \text{ sq. units.}$$

Example 7. *Find the area enclosed by the circle $x^2 + y^2 = a^2$.* (*NCERT*)

Solution. The given circle is $x^2 + y^2 = a^2$. The area enclosed by the given circle is shown shaded in fig. 8.13. Since the area is symmetrical about both the axes,

required area = 4 (area of the region bounded by the arc of the circle in first quadrant, the x-axis and the lines $x = 0$ and $x = a$)

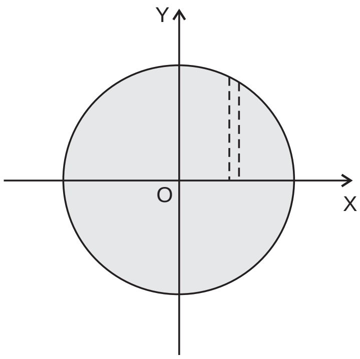


Fig. 8.13.

$$= 4\int_0^a y\,dx = 4\int_0^a \sqrt{a^2 - x^2}\,dx$$

($\because$ $x^2 + y^2 = a^2 \Rightarrow y = \sqrt{a^2 - x^2}$ in the first quadrant)

$$= 4\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\left[\left(0 + \frac{a^2}{2}\sin^{-1} 1\right) - \left(0 + \frac{a^2}{2}\sin^{-1} 0\right)\right] = 4\left[\frac{a^2}{2}\cdot\frac{\pi}{2} - 0\right] = \pi a^2 \text{ sq. units.}$$

Example 8. *Draw a rough sketch of the graph of the function $y = 2\sqrt{1 - x^2}$, $x \in [0, 1]$ and evaluate the area enclosed between the curve and the axes.*

Solution. The given curve is $y = 2\sqrt{1 - x^2}$

$\Rightarrow$ $\frac{y^2}{4} = 1 - x^2 \Rightarrow \frac{x^2}{1} + \frac{y^2}{4} = 1$, which represents an ellipse of the second standard form. Hence, the given equation $y = 2\sqrt{1 - x^2}$ represents the portion of the ellipse lying in the first quadrant. Its rough sketch is shown in fig. 8.14.

The required area = the area of the shaded region

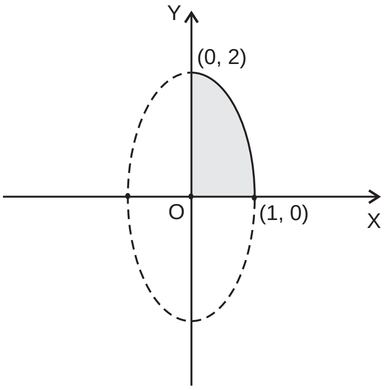


Fig. 8.14.

$$= \int_0^1 y\,dx = \int_0^1 2\sqrt{1 - x^2}\,dx$$

$$= 2\left[\frac{x\sqrt{1 - x^2}}{2} + \frac{1}{2}\sin^{-1} x\right]_0^1 = \left[x\sqrt{1 - x^2} + \sin^{-1} x\right]_0^1$$

$$= (0 + \sin^{-1} 1) - (0 + \sin^{-1} 0) = \frac{\pi}{2} \text{ sq. units.}$$

Example 9. *Find the area bounded by the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *and the ordinates* $x = 0$ *and* $x = ae$ *where* $b^2 = a^2 (1 - e^2)$ *and* $0 < e < 1$. (*NCERT*)

Solution. The given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \quad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow \quad y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

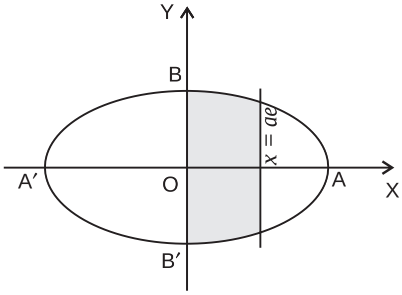


Fig. 8.15.

The required area is shown shaded in fig. 8.15.

Since the area is symmetrical about the x-axis,

required area = 2 (area of the region bounded by the given ellipse, x-axis and the lines $x = 0$ and $x = ae$)

$$= 2\int_0^{ae} y\,dx = 2\int_0^{ae} \frac{b}{a}\sqrt{a^2 - x^2}\,dx \qquad (\because\ y \geq 0 \text{ in the first quadrant})$$

$$= 2\,\frac{b}{a}\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^{ae}$$

$$= \frac{b}{a}\left[\left(ae\sqrt{a^2 - a^2 e^2} + a^2 \sin^{-1} e\right) - \left(0 + \frac{a^2}{2}\sin^{-1} 0\right)\right]$$

$$= ab\left(e\sqrt{1 - e^2} + \sin^{-1} e\right) \text{ sq. units.}$$

Example 10. *The area between* $x = y^2$ *and* $x = 4$ *is divided into two equal parts by the line* $x = a$, *find the value of* a. (*NCERT*)

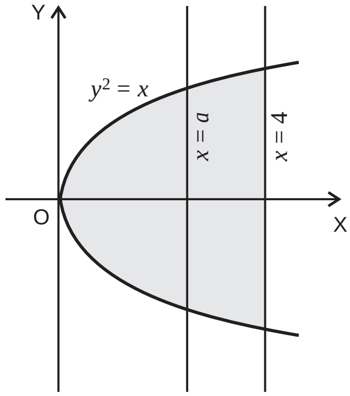


Fig. 8.16.

Solution. The given curve is $y^2 = x$ which represents a right hand parabola with vertex (0, 0).

The area bounded by the parabola and the line $x = 4$ is shown shaded in fig. 8.16.

$$\text{This area} = 2\int_0^4 y\,dx = 2\int_0^4 \sqrt{x}\,dx$$

$$= 2\,.\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^4 = \frac{4}{3}\left[4^{3/2} - 0\right] = \frac{4}{3}[8 - 0] = \frac{32}{3}.$$

Since the line $x = a$ divides this area into two equal parts, therefore,

$$2\int_0^a \sqrt{x}\,dx = \frac{1}{2}.\frac{32}{3} \Rightarrow \int_0^a \sqrt{x}\,dx = \frac{8}{3}$$

$$\Rightarrow \quad \left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^a = \frac{8}{3} \Rightarrow \frac{2}{3}\left[a^{3/2} - 0\right] = \frac{8}{3}$$

$$\Rightarrow \quad a^{3/2} = 4 \Rightarrow a = 4^{2/3}$$

$$\Rightarrow \quad a = \sqrt[3]{16}.$$

Example 11. *Find the area of the smaller region bounded by the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *and the straight line* $\frac{x}{a}+\frac{y}{b}=1$. *(C.B.S.E. 2013, 04)*

Solution. The given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{y^2}{b^2}=1-\frac{x^2}{a^2}$

$\Rightarrow \quad y = \frac{b}{a}\sqrt{a^2-x^2}$ (∵ In first quadrant, $y \geq 0$)

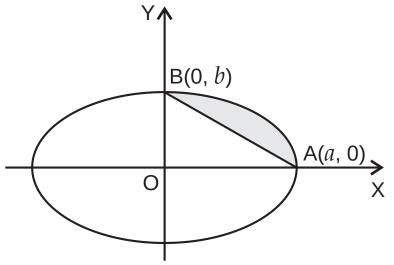


Fig. 8.17.

The given line is $\frac{x}{a}+\frac{y}{b}=1$

$\Rightarrow \quad \frac{y}{b} = 1-\frac{x}{a} = \frac{a-x}{a}$

$\Rightarrow \quad y = \frac{b}{a}(a-x)$

The area of the smaller region bounded by the given ellipse and the given line is shown shaded in fig. 8.17.

Required area $= \int_0^a \left(\frac{b}{a}\sqrt{a^2-x^2}-\frac{b}{a}(a-x)\right)dx$ (Article 8.2)

$$= \frac{b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}-ax+\frac{x^2}{2}\right]_0^a$$

$$= \frac{b}{a}\left[\left(0+\frac{a^2}{2}\sin^{-1}1-a^2+\frac{a^2}{2}\right)-\left(0+\frac{a^2}{2}\sin^{-1}0-0+0\right)\right]$$

$$= \frac{b}{a}\left[\left(\frac{a^2}{2}\cdot\frac{\pi}{2}-\frac{a^2}{2}\right)-0\right] = \frac{1}{4}(\pi-2)\,ab \text{ sq. units.}$$

Example 12. *Find the area of the region included between the curve* $4y = 3x^2$ *and the line* $2y = 3x + 12$. *(C.B.S.E. 2013)*

Solution. The given curve is $4y = 3x^2$ ...(*i*)

It can be written as $y = \frac{3}{4}x^2$, which represents an upward parabola with vertex at (0, 0).

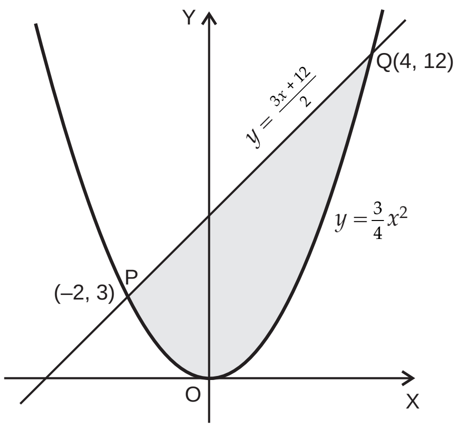


Fig. 8.18.

The given line is $2y = 3x + 12$

$\Rightarrow \quad y = \frac{3x+12}{2}$ ...(*ii*)

Solving (*i*) and (*ii*), we get

$$\frac{3x+12}{2} = \frac{3}{4}x^2$$

$\Rightarrow \quad 6x + 24 = 3x^2$

$\Rightarrow \quad x^2 - 2x - 8 = 0 \Rightarrow (x+2)(x-4) = 0$

$\Rightarrow \quad x = -2,\ x = 4.$

∴ The points of intersection are P (–2, 3) and Q (4, 12).

∴ Required area = area of the shaded region

$$= \int_{-2}^{4}\left(\frac{3x+12}{2}-\frac{3}{4}x^2\right)dx$$ (Article 8.2)

$$= \left[\frac{3}{2}.\frac{x^2}{2} + 6x - \frac{3}{4}.\frac{x^3}{3}\right]_{-2}^{4} = \frac{1}{4}\left[3x^2 + 24x - x^3\right]_{-2}^{4}$$

$$= \frac{1}{4}\left[(48 + 96 - 64) - (12 - 48 + 8)\right]$$

$$= \frac{1}{4}.108 = 27 \text{ sq. units.}$$

Example 13. *Find the area enclosed by the curve $y = -x^2$ and the straight line $x + y + 2 = 0$.* (*NCERT Examplar Problems*)

Solution. The given curve is $y = -x^2$ ...(*i*)

It represents a downward parabola with vertex O(0, 0).

The given line is $x + y + 2 = 0$

$\Rightarrow \quad y = -(x + 2)$ ...(*ii*)

Solving (*i*) and (*ii*), we get

$-x^2 = -(x + 2)$

$\Rightarrow \quad x^2 - x - 2 = 0$

$\Rightarrow \quad (x + 1)(x - 2) = 0 \Rightarrow x = -1, 2.$

When $x = -1$, $y = -1$ and when $x = 2$, $y = -4$.

$\therefore$ The points of intersection are P (–1, –1) and Q (2, –4).

The required area is shown shaded in fig. 8.19. We note that the required area lies below x-axis, therefore,

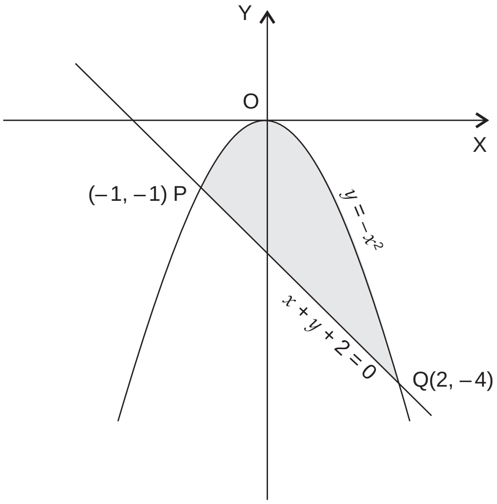


Fig. 8.19.

$$\therefore \quad \text{required area} = \left|\int_{-1}^{2} (-(x + 2) - (-x^2))\,dx\right| \qquad \text{(Article 8.2)}$$

$$= \left|\left[-\left(\frac{x^2}{2} + 2x\right) + \frac{x^3}{3}\right]_{-1}^{2}\right|$$

$$= \left|\left(-6 + \frac{8}{3}\right) - \left(-\left(\frac{1}{2} - 2\right) - \frac{1}{3}\right)\right| = \frac{9}{2} \text{ sq. units.}$$

Example 14. *Find the area enclosed by the parabola $y^2 = x$ and the line $y + x = 2$.* (*C.B.S.E. 2009, 05*)

Solution. The given parabola is $y^2 = x$ ...(*i*)

It represents a right hand parabola with vertex at (0, 0).

The given line is $y + x = 2$

i.e. $\quad x = 2 - y$ ...(*ii*)

Solving (*i*) and (*ii*), we get

$y^2 = 2 - y \Rightarrow y^2 + y - 2 = 0$

$\Rightarrow \quad (y - 1)(y + 2) = 0 \Rightarrow y = 1, -2$

When $y = 1$, $x = 1$, when $y = -2$, $x = 4$

The points of intersection are P (1, 1) and Q (4, –2).

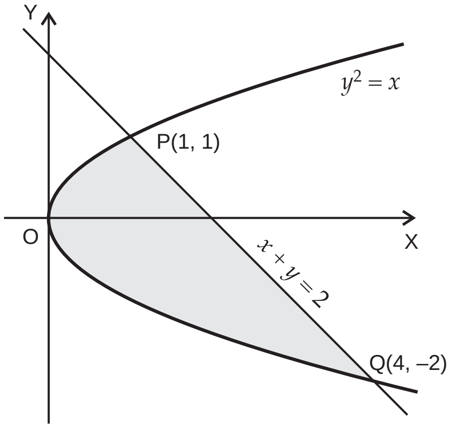


Fig. 8.20.

The required area = area of the shaded region

$$= \int_{-2}^{1} ((2-y)-y^2)\,dy = \left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-2}^{1}$$

$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right)$$

$$= 2 - \frac{1}{2} - \frac{1}{3} + 6 - \frac{8}{3} = 4\frac{1}{2} \text{ sq. units.}$$

Example 15. *Find the area bounded by the curve $y = \sqrt{x}$ and the line $x = 2y + 3$ in the first quadrant and the x-axis.* (*NCERT Examplar Problems*)

Solution. Given $y = \sqrt{x}$ (Note that $y \ge 0$)

$\Rightarrow \quad y^2 = x, y \ge 0$ ...(*i*)

It represents a part of a right hand parabola with vertex (0, 0).

Given line is $x = 2y + 3$ ...(*ii*)

Solving (*i*) and (*ii*), we get

$y^2 = 2y + 3 \Rightarrow y^2 - 2y - 3 = 0$

$\Rightarrow (y+1)(y-3) = 0$

$\Rightarrow y = -1, 3$ but $y \ge 0$

$\Rightarrow y = 3.$

When $y = 3$, from (*ii*), $x = 9$.

Hence, the curve and the line meet at the point P(9, 3).

Required area is shown shaded in fig. 8.21.

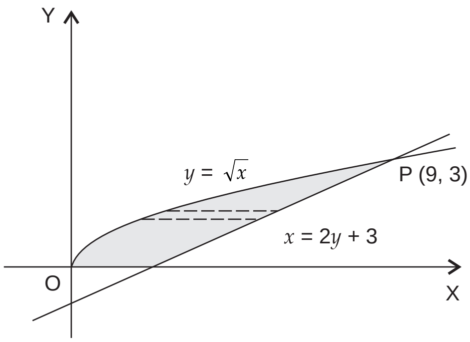


Fig. 8.21.

$$\text{Required area} = \int_{0}^{3} ((2y+3)-y^2)\,dy \qquad \text{(Article 8.2)}$$

$$= \left[y^2 + 3y - \frac{y^3}{3}\right]_0^3 = (9 + 9 - 9) - (0 + 0 - 0)$$

$$= 9 \text{ sq. units.}$$

Example 16. *Using integration, find the area of the triangular region whose vertices are (–1, 1), (0, 5) and (3, 2).* (*C.B.S.E. 2008*)

Solution. Let the vertices of the given triangle be A (–1, 1), B (0, 5) and C (3, 2), shown in fig. 8.22.

The equations of the sides AB, BC and CA are

$y = 4x + 5$ ...(*i*)

$y = -x + 5$ ...(*ii*)

and $y = \frac{1}{4}x + \frac{5}{4}$ ...(*iii*)

respectively. (Obtain these equations)

Required area = area of the region bounded by ΔABC = area of the region bounded by ΔBAD + area of the region bounded by ΔBDC

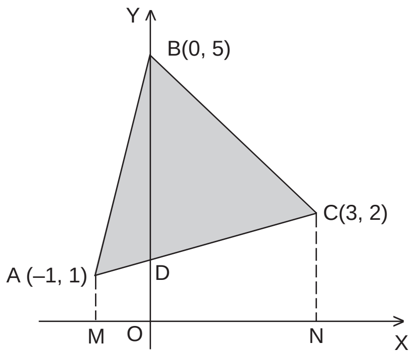


Fig. 8.22.

$$\int_{-1}^{0} \left((4x+5) - \left(\frac{1}{4}x + \frac{5}{4}\right)\right) dx + \int_{0}^{3} \left((-x+5) - \left(\frac{1}{4}x + \frac{5}{4}\right)\right) dx \qquad \text{(Article 8.2)}$$

$$= \int_{-1}^{0} \left(\frac{15}{4}x + \frac{15}{4}\right) dx + \int_{0}^{3} \left(-\frac{5}{4}x + \frac{15}{4}\right) dx = \frac{15}{4}\int_{-1}^{0} (x+1)\,dx - \frac{5}{4}\int_{0}^{3} (x-3)\,dx$$

$$= \frac{15}{4}\left[\frac{x^2}{2} + x\right]_{-1}^{0} - \frac{5}{4}\left[\frac{x^2}{2} - 3x\right]_{0}^{3} = \frac{15}{4}\left[0 - \left(\frac{1}{2} - 1\right)\right] - \frac{5}{4}\left[\left(\frac{9}{2} - 9\right) - 0\right]$$

$$= \frac{15}{8} + \frac{45}{8} = \frac{60}{8} = \frac{15}{2} \text{ sq. units.}$$

Example 17. *Using the method of integration, find the area of the region bounded by the lines* $2x + y = 4$, $3x - 2y = 6$ *and* $x - 3y + 5 = 0$. (*C.B.S.E. 2011, 09*)

Solution. The equations of the given lines are :

$2x + y = 4$ *i.e.* $y = 2(2 - x)$ ...(*i*)

$3x - 2y = 6$ *i.e.* $y = \frac{3}{2}(x - 2)$ ...(*ii*)

$x - 3y + 5 = 0$ *i.e.* $y = \frac{1}{3}(x + 5)$ ...(*iii*)

On solving (*i*) and (*ii*) simultaneously, we get $x = 2$, $y = 0$

∴ The point of intersection of (*i*) and (*ii*) is A(2, 0).

On solving (*ii*) and (*iii*) simultaneously, we get $x = 4$, $y = 3$.

∴ The point of intersection of (*ii*) and (*iii*) is B(4, 3).

On solving (*iii*) and (*i*) simultaneously, we get $x = 1$, $y = 2$.

∴ The point of intersection of (*iii*) and (*i*) is C(1, 2).

The area of the region bounded by the given lines is shown shaded in fig. 8.23. From C, draw CM ⊥ OX and from B, draw BN ⊥ OX.

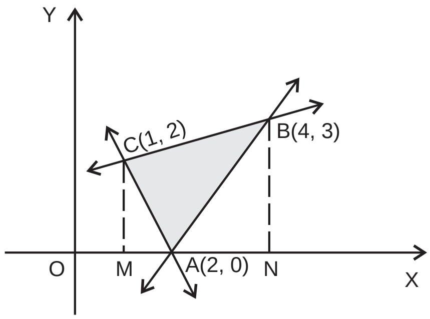


Fig. 8.23.

Required area = area of region CMNB – area of ΔCMA – area of ΔBAN

$$= \int_{1}^{4} \frac{1}{3}(x+5)\,dx - \int_{1}^{2} 2(2-x)\,dx - \int_{1}^{4} \frac{3}{2}(x-2)\,dx$$

$$= \frac{1}{3}\cdot\left[\frac{(x+5)^2}{2}\right]_{1}^{4} - 2\cdot\left[\frac{(2-x)^2}{2.(-1)}\right]_{1}^{2} - \frac{3}{2}\cdot\left[\frac{(x-2)^2}{2}\right]_{2}^{4}$$

$$= \frac{1}{6}(81 - 36) + (0 - 1) - \frac{3}{4}(4 - 0)$$

$$= \frac{45}{6} - 1 - 3 = \frac{15}{2} - 4 = \frac{7}{2}.$$

Hence, the required area $= \frac{7}{2}$ sq. units.

Example 18. *Find the area of the region bounded by the parabolas* $y^2 = 4ax$ *and* $x^2 = 4ay$, $a > 0$. (*C.B.S.E. 2013, 09, 08*)

Solution. The given parabolas are

$y^2 = 4ax$ ...(*i*)

$x^2 = 4ax$ ...(*ii*)

The equation (*i*) represents a right hand parabola with vertex (0, 0) and the equation (*ii*) represents an upward parabola with vertex (0, 0).

Solving (*i*) and (*ii*) simultaneously, we get

$$\left(\frac{x^2}{4a}\right)^2 = 4ax \Rightarrow x^4 - 64a^3x = 0$$

$\Rightarrow \quad x(x^3 - 64a^3) = 0 \Rightarrow x = 0, 4a.$

When $x = 0$, $y = 0$ and when $x = 4a$, $y = 4a$.

Thus, the points of intersection of the given parabolas are O(0, 0) and P(4*a*, 4*a*).

The area bounded by the given curves is shown shaded in fig. 8.24.

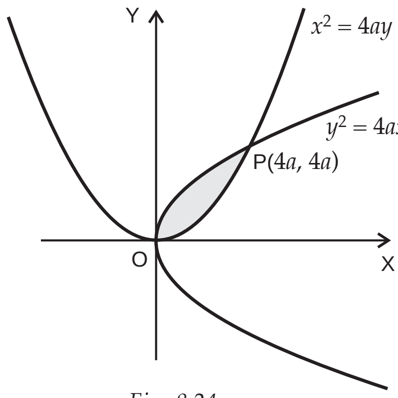


Fig. 8.24.

The requirede area = area of the shded region

$$= \int_0^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) dx$$

$(\because\ y^2 = 4ax \Rightarrow y = 2\sqrt{ax}$ in the first quad. and $x^2 = 4ay \Rightarrow y = \frac{x^2}{4a})$

$$= 2\sqrt{ax}\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^{4a} - \frac{1}{4a}\cdot\left[\frac{x^3}{3}\right]_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}\,((4a)^{3/2} - 0) - \frac{1}{12a}(64a^3 - 0)$$

$$= \frac{4}{3}\sqrt{a}\,8a^{3/2} - \frac{16}{3}a^2 = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2.$$

Hence, the required area $= \frac{16}{3}a^2$ sq. units.

Example 19. *Using integration, find the area of the region bounded by the following curves, after making a rough sketch :*

$y = 1 + |x + 1|, x = -3, x = 3, y = 0.$ (*NCERT Examplar Problems*)

Solution. Given $y = 1 + |x + 1| = \begin{cases} 1 + x + 1, & \text{if} \quad x + 1 \geq 0 \\ 1 - (x + 1), & \text{if} \quad x + 1 < 0 \end{cases}$

$$\Rightarrow \quad y = \begin{cases} x + 2, & \text{if} \quad x \geq -1 \\ -x, & \text{if} \quad x < -1 \end{cases}$$

and $x = -3$, $x = 3$, $y = 0$.

The required area is shown shaded in fig. 8.25.

$\therefore$ The required area = the area of the shaded region

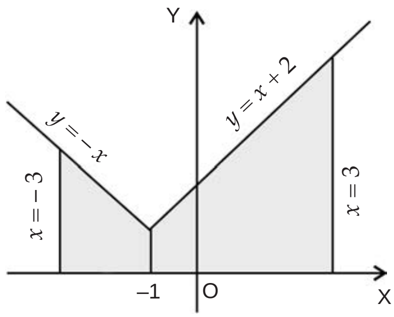


Fig. 8.25.

$$= \int_{-3}^{-1} (-x)\,dx + \int_{-1}^{3} (x + 2)\,dx$$

$$= -\left[\frac{x^2}{2}\right]_{-3}^{-1} + \left[\frac{x^2}{2} + 2x\right]_{-1}^{3}$$

$$= -\frac{1}{2}(1 - 9) + \left[\left(\frac{9}{2} + 6\right) - \left(\frac{1}{2} - 2\right)\right]$$

$= 4 + 12 = 16$ sq. units.

Example 20. *Find the area of the region bounded by $y = x^2$ and $y = |x|$.* (*C.B.S.E. 2013, 08 C*)

Solution. The equation $y = x^2$ represents an upward parabola with vertex at (0, 0).

$$y = |x| = \begin{cases} x & , x \ge 0 \\ -x & , x < 0 \end{cases}$$

which represents parts of two straight lines.

The line $y = x$, $x \ge 0$, meets the parabola $y = x^2$ where $x = x^2$ *i.e.* where $x = 0, 1$.

When $x = 0$, $y = 0$; when $x = 1$, $y = 1$.

$\therefore$ The points of intersection of the line $y = x$ $(x \ge 0)$ and the parabola $y = x^2$ are (0, 0) and (1, 1).

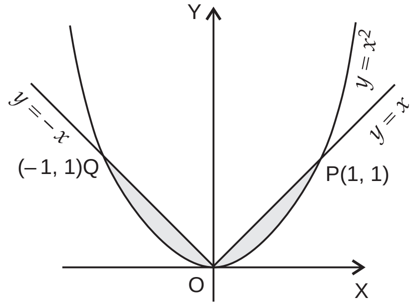


Fig. 8.26.

The line $y = -x$, $x < 0$ meets the parabola $y = x^2$ where $-x = x^2$ *i.e.* where $x = -1$.

When $x = -1$, $y = 1$.

$\therefore$ The point of intersection of the line $y = -x$, $x < 0$ and the parabola $y = x^2$ is (–1, 1).

The required area is shown shaded in fig. 8.26.

$$\text{Required area} = 2\int_0^1 (x - x^2)\, dx \qquad \text{(by symmetry)}$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(\left(\frac{1}{2} - \frac{1}{3}\right) - (0 - 0)\right) \text{ sq. units} = \frac{1}{3} \text{ sq. units.}$$

Example 21. *Find the area of the region in the first quadrant enclosed by the x-axis, the line $x = \sqrt{3}y$ and the circle $x^2 + y^2 = 4$.* (*C.B.S.E. 2012*)

Solution. The given circle is $x^2 + y^2 = 4$ ...(*i*)

Its centre is (0, 0) and radius = 2.

The given line is $x = \sqrt{3}y$ ...(*ii*)

Solving (*i*) and (*ii*), we get

$3y^2 + y^2 = 4 \Rightarrow 4y^2 = 4 \Rightarrow y^2 = 1$

$\Rightarrow$ $y = 1, -1$, but in first quadrant $y \ge 0$

$\Rightarrow$ $y = 1$.

When $y = 1$, $x = \sqrt{3}\,.1 = \sqrt{3}$.

$\therefore$ The point of intersection in the first quadrant is P $(\sqrt{3}, 1)$.

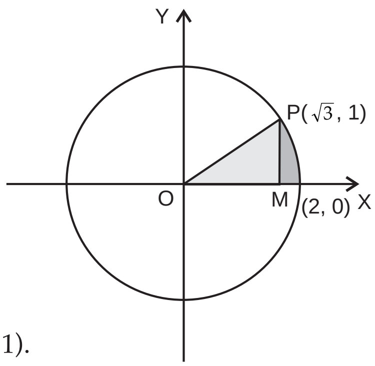


Fig. 8.27.

$\therefore$ The required area = the area of the shaded region

$$= \int_0^{\sqrt{3}} \frac{x}{\sqrt{3}}\, dx + \int_{\sqrt{3}}^{2} \sqrt{4 - x^2}\, dx$$

$$= \frac{1}{\sqrt{3}}\left[\frac{x^2}{2}\right]_0^{\sqrt{3}} + \left[\frac{x\sqrt{4 - x^2}}{2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$

$$= \frac{1}{\sqrt{3}}\left(\frac{3}{2} - 0\right) + \frac{1}{2}(0 - \sqrt{3}\,.1) + 2\left(\sin^{-1} 1 - \sin^{-1}\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} + 2\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{\pi}{3} \text{ sq. units.}$$

Example 22. *Find the area of the region enclosed between the two circles $x^2 + y^2 = 9$ and $(x - 3)^2 + y^2 = 9$.* (*C.B.S.E. 2009*)

Solution. The given circles are

$$x^2 + y^2 = 9 \qquad \ldots(i)$$

and $$(x - 3)^2 + y^2 = 9 \qquad \ldots(ii)$$

The equation (i) represents a circle with centre (0, 0) and radius 3 units and the equation (ii) represents a circle with centre (3, 0) and radius 3 units.

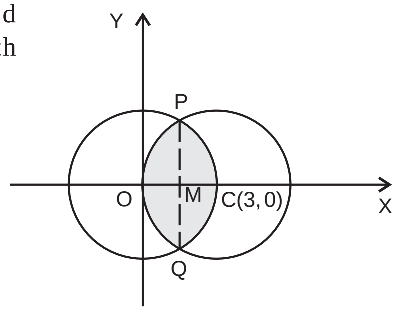


Fig. 8.28.

Solving (i) and (ii) simultaneously, we get

$$x^2 - (x - 3)^2 = 0 \Rightarrow 6x - 9 = 0 \Rightarrow x = \frac{3}{2}.$$

When $x = \frac{3}{2}$, from (i), we get

$$\left(\frac{3}{2}\right)^2 + y^2 = 9 \Rightarrow y^2 = 9 - \frac{9}{4} = \frac{27}{4} \Rightarrow y = \pm\frac{3\sqrt{3}}{2}.$$

$\therefore$ The points of intersection of (i) and (ii) are

$$P\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right),\ Q\left(\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right).$$

The area of the region enclosed between the given circles is shown shaded in fig. 8.28.

Note that the required area is symmetrical about x-axis. As the radii of the two circles are equal, the required area is also symmetrical about the line PQ.

From (i), $y = \sqrt{9 - x^2}$ (in the first quadrant)

$\therefore$ Required area = area of the shaded region

= 2 (area of the region OMCPO) = 2 × 2 (area of region MCPM)

$$= 4\int_{3/2}^{3} \sqrt{9 - x^2}\, dx$$

$$= 4\left[\frac{x\sqrt{9 - x^2}}{2} + \frac{3^2}{2}\sin^{-1}\frac{x}{3}\right]_{3/2}^{3}$$

$$= 2\left((0 + 9\sin^{-1}(1)) - \left(\frac{3}{2}\sqrt{9 - \frac{9}{4}} + 9\sin^{-1}\frac{1}{2}\right)\right)$$

$$= 2\left(9 \times \frac{\pi}{2} - \frac{9\sqrt{3}}{4} - 9 \times \frac{\pi}{6}\right) = 6\pi - \frac{9\sqrt{3}}{2}.$$

Hence, the required area = $\left(6\pi - \frac{9}{2}\sqrt{3}\right)$ sq. units.

Example 23. *Find the smaller of the two areas in which the circle $x^2 + y^2 = 4$ is divided by the parabola $y^2 = 3(2x - 1)$.*

Solution. The given circle is $x^2 + y^2 = 4$ $\ldots(i)$

Its centre is (0, 0) and radius = 2.

The given parabola is $y^2 = 3(2x - 1)$ $\ldots(ii)$

$\Rightarrow y^2 = 6\left(x - \frac{1}{2}\right)$, which is a right hand parabola with vertex at $\left(\frac{1}{2}, 0\right)$.

Solving (i) and (ii), we get

$x^2 + 3(2x - 1) = 4 \Rightarrow x^2 + 6x - 7 = 0$

$\Rightarrow \quad (x + 7)(x - 1) = 0 \Rightarrow x = -7, 1$

but $\quad x \geq \frac{1}{2} \quad \Rightarrow \quad x = 1$

When $x = 1$, $y^2 = 3 \Rightarrow y = \pm\sqrt{3}$.

$\therefore$ The points of intersection of (i) and (ii) are P$(1, \sqrt{3})$, Q$(1, -\sqrt{3})$.

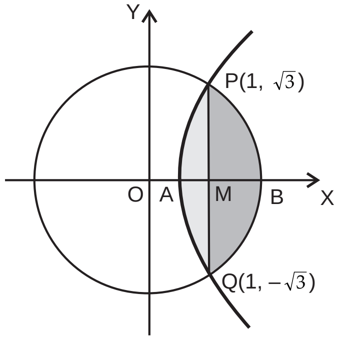


Fig. 8.29.

$\therefore$ The required area = the area of the shaded region

= 2 . area of the region PAB

(by symmetry)

Required area = 2 (area of the region PAM + area of the region PMB)

$$= 2\int_{1/2}^{1}\sqrt{3(2x-1)}\,dx + 2\int_{1}^{2}\sqrt{4-x^2}\,dx$$

$$= 2.\sqrt{3}\left[\frac{(2x-1)^{3/2}}{\frac{3}{2}.2}\right]_{1/2}^{1} + 2.\left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= \frac{2\sqrt{3}}{3}(1-0) + (0 + 4\sin^{-1}(1)) - \left(\sqrt{3} + 4\sin^{-1}\left(\frac{1}{2}\right)\right) = \frac{2\sqrt{3}}{3} + 4.\frac{\pi}{2} - \left(\sqrt{3} + 4.\frac{\pi}{6}\right)$$

$$= \sqrt{3}\left(\frac{2}{3} - 1\right) + 2\pi - \frac{2}{3}\pi = -\frac{\sqrt{3}}{3} + \frac{4\pi}{3} = \frac{1}{3}(4\pi - \sqrt{3}) \text{ sq. units.}$$

Example 24. *Find the area of that part of the circle $x^2 + y^2 = 16$ which is exterior to the parabola $y^2 = 6x$.* (*C.B.S.E. 2008, 07*)

Solution. The given circle is $x^2 + y^2 = 16$ ...(i)

Its centre is (0, 0) and radius = 4.

The given parabola is $y^2 = 6x$ ...(ii)

which is a right hand parabola with vertex at (0, 0).

Solving (i) and (ii), we get

$x^2 + 6x - 16 = 0$

$\Rightarrow \quad (x - 2)(x + 8) = 0$

$\Rightarrow \quad x = 2, -8$ but $x \geq 0$

$\Rightarrow \quad x = 2.$

When $x = 2$, $y^2 = 6.2 = 12 \Rightarrow y = \pm 2\sqrt{3}$.

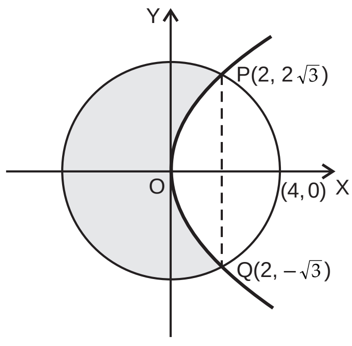


Fig. 8.30.

$\therefore$ The points of intersection of (i) and (ii) are P$(2, 2\sqrt{3})$ and Q$(2, -2\sqrt{3})$.

$\therefore$ The required area = the area of the shaded region

= area of the circle – area of the circle interior to the parabola

$$= 4\int_0^4 \sqrt{16-x^2}\,dx - 2\left[\int_0^2 \sqrt{6x}\,dx + \int_2^4 \sqrt{16-x^2}\,dx\right] \qquad \text{(by symmetry)}$$

$$= 4\left[\frac{x\sqrt{16-x^2}}{2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_0^4 - 2\sqrt{6}\left[\frac{x^{3/2}}{3/2}\right]_0^2 - 2\left[\frac{x\sqrt{16-x^2}}{2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_2^4$$

$$= 4\left[\frac{1}{2}(0-0)+8(\sin^{-1}1-\sin^{-1}0)\right]-\frac{4\sqrt{6}}{3}\left[2\sqrt{2}-0\right]$$

$$-\,2\left[\frac{1}{2}(0-4\sqrt{3})+8\left(\sin^{-1}1-\sin^{-1}\frac{1}{2}\right)\right]$$

$$= 32\left(\frac{\pi}{2}-0\right)-\frac{16}{\sqrt{3}}+4\sqrt{3}-16\left(\frac{\pi}{2}-\frac{\pi}{6}\right)$$

$$= 16\pi-\frac{16}{3}\pi-\frac{4}{\sqrt{3}}=\frac{32}{3}\pi-\frac{4}{3}\sqrt{3} = \frac{4}{3}\ (8\pi-\sqrt{3}) \text{ sq. units.}$$

Example 25. *Find the area of the region above the x-axis, included between the curves $y^2 = ax$ and $x^2 + y^2 = 2ax$.* (*NCERT Examplar Problems*)

Solution. The two curves are $y^2 = ax$ ...(*i*)

and $x^2 + y^2 - 2ax = 0$ *i.e.* $(x-a)^2 + y^2 = a^2$ ...(*ii*)

The curve (*i*) is a right hand parabola with vertex at (0, 0) and the curve (*ii*) is a circle with centre $(a, 0)$ and radius a.

Solving (*i*) and (*ii*) simultaneously, we get $(x-a)^2 + ax = a^2$

$\Rightarrow\ x^2 - ax = 0 \ \Rightarrow\ x(x-a) = 0$

$\Rightarrow\ x = 0, a.$

When $x = 0,\ y^2 = 0 \ \Rightarrow\ y = 0.$

When $x = a,\ y^2 = a\,.\,a = a^2 \ \Rightarrow\ y = a, -a.$

The points of intersection of (*i*) and (*ii*) are (0, 0), (a, a) and $(a, -a)$.

The required area is shown shaded in fig. 8.31.

Above the x-axis, the arc of the circle from O to A is a part of the curve $y = \sqrt{2ax - x^2}$ and the arc of the parabola is a part of the curve $y = \sqrt{ax}$.

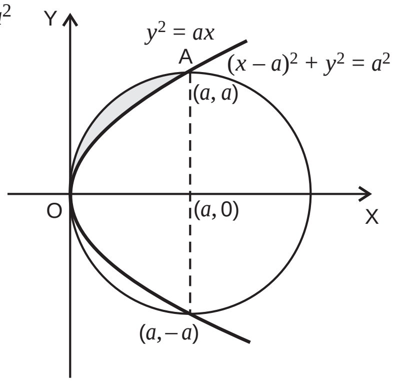


Fig. 8.31.

Hence, the required area $= \displaystyle\int_0^a (\sqrt{2ax - x^2} - \sqrt{ax})\,dx = \int_0^a (\sqrt{a^2-(x-a)^2} - \sqrt{a}\,\sqrt{x})\,dx$

$$= \left[\frac{x-a}{2}\sqrt{a^2-(x-a)^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)-\sqrt{a}\,\frac{x^{3/2}}{3/2}\right]_0^a$$

$$= \left(0+\frac{a^2}{2}\sin^{-1}0-\frac{2}{3}\sqrt{a}\,.\,a^{3/2}\right)-\left(0+\frac{a^2}{2}\sin^{-1}(-1)-0\right)$$

$$= -\frac{2}{3}a^2-\frac{a^2}{2}\left(-\frac{\pi}{2}\right)=\frac{-8a^2+3\pi a^2}{12}=\frac{(3\pi-8)a^2}{12} \text{ sq. units.}$$

Example 26. *Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$.* (*C.B.S.E. 2010*)

Solution. Given equations are

$$4x^2 + 4y^2 = 9 \ \textit{i.e.}\ x^2 + y^2 = \left(\frac{3}{2}\right)^2 \qquad \text{...}(i)$$

and $x^2 = 4y$ ...(*ii*)

The equation (*i*) represents a circle with centre (0, 0) and radius $\frac{3}{2}$ and the equation (*ii*) represents an upward parabola with vertex (0, 0).

Solving (*i*) and (*ii*) simultaneously, we get

$$16y + 4y^2 = 9 \Rightarrow 4y^2 - 16y - 9 = 0$$

$$\Rightarrow\quad (2y-1)(2y+9) = 0$$

$\Rightarrow \quad y = \frac{1}{2}, -\frac{9}{2}$ but $y = -\frac{9}{2}$ does not give real values of x

$\Rightarrow \quad y = \frac{1}{2}$.

When $y = \frac{1}{2}$, from (*ii*), $x^2 = 2 \Rightarrow x = \sqrt{2}, -\sqrt{2}$.

$\therefore$ The points of intersection of (*i*) and (*ii*) are

$P\left(\sqrt{2}, \frac{1}{2}\right)$ $Q\left(-\sqrt{2}, \frac{1}{2}\right)$.

The circle meets y-axis at B when OB = $\frac{3}{2}$.

The area interior to the circle (*i*) and parabola (*ii*) is shown shaded in fig. 8.32.

Note that the required area is symmetrical about y-axis.

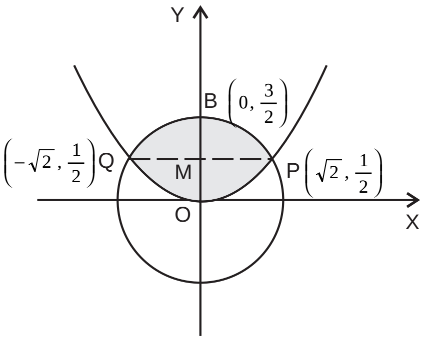


Fig. 8.32.

From (*i*), $x = \sqrt{\left(\frac{3}{2}\right)^2 - y^2}$ and from (*ii*), $x = 2\sqrt{y}$ (in the first quadrant)

$\therefore$ Required area = area of the shaded region

= 2 (area of the region OPBO)

= 2 (area of region OPMO + area of region MPBM)

$$= 2\int_0^{1/2} 2\sqrt{y}\, dy + 2\int_{1/2}^{3/2} \sqrt{\left(\frac{3}{2}\right)^2 - y^2}\, dy$$

$$= 4\left[\frac{y^{3/2}}{\frac{3}{2}}\right]_0^{1/2} + \left[\frac{y\sqrt{\left(\frac{3}{2}\right)^2 - y^2}}{2} + \frac{\left(\frac{3}{2}\right)^2}{2}\sin^{-1}\left(\frac{y}{\frac{3}{2}}\right)\right]_{1/2}^{3/2}$$

$$= \frac{8}{3}\left(\left(\frac{1}{2}\right)^{3/2} - 0\right) + \left(0 - \frac{1}{2}\sqrt{\frac{9}{4} - \frac{1}{4}}\right) + \frac{9}{4}\left(\sin^{-1} 1 - \sin^{-1}\frac{1}{3}\right)$$

$$= \frac{8}{3} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2}\sqrt{2} + \frac{9}{4}\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right)$$

$$= \frac{4}{3\sqrt{2}} - \frac{1}{\sqrt{2}} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3} = \frac{1}{3\sqrt{2}} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3}.$$

Hence, the required area = $\left(\frac{1}{3\sqrt{2}} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3}\right)$ sq. units.

Example 27. *Sketch the graph of* $f(x) = \begin{cases} |x-2|+2, & x \le 2 \\ x^2 - 2, & x > 2 \end{cases}$. *Evaluate* $\int_0^4 f(x)\, dx$. *What does the value of this integral represent on the graph?* (*C.B.S.E. Sample Paper*)

Solution. Given $y = f(x) = \begin{cases} |x-2|+2, & x \le 2 \\ x^2 - 2, & x > 2. \end{cases}$

When $x \le 2$, $y = |x - 2| + 2 = -(x - 2) + 2 = 4 - x$

i.e. when $x \le 2$, $y = 4 - x$, which represents a part of a straight line.

Table of values

x	0	2
y	4	2

When $x > 2$, $y = x^2 - 2$ *i.e.* $y + 2 = x^2$, which represents a part of an upward parabola with vertex at $(0, -2)$.

Table of values

x	3	4
y	7	14

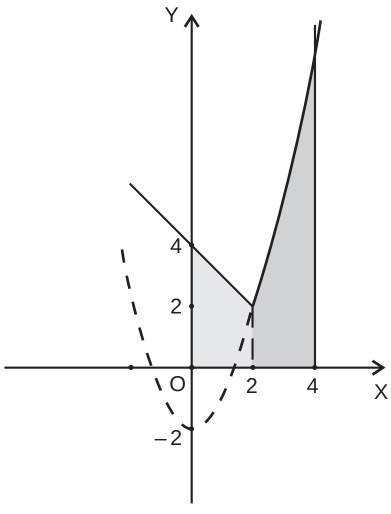


Fig. 8.33.

A rough sketch of the graph of the given function is shown in fig. 8.33.

$$\int_0^4 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx$$

$$= \int_0^2 (4 - x)\,dx + \int_2^4 (x^2 - 2)\,dx$$

$$= \left[4x - \frac{x^2}{2}\right]_0^2 + \left[\frac{x^3}{3} - 2x\right]_2^4$$

$$= (8 - 2) - (0 - 0) + \left(\frac{64}{3} - 8\right) - \left(\frac{8}{3} - 4\right)$$

$$= 6 + \frac{56}{3} - 4 = \frac{62}{3} \text{ sq. units.}$$

The value of $\int_0^4 f(x)\,dx$ on the graph represents the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = 0$ and $x = 4$ *i.e.* the area of the shaded region.

Example 28. *Make a rough sketch of the region given below and find its area using methods of integration :* $\{(x, y) ; 0 \le y \le x^2 + 3, 0 \le y \le 2x + 3, 0 \le x \le 3\}$.

Solution. First, we note that the curve $y = x^2 + 3$ is an upward parabola with vertex at (0, 3). The curve $y = 2x + 3$ is a straight line. The required area lies below the parabola as well as the line $y = 2x + 3$ and above the x-axis between the ordinates $x = 0$ and $x = 3$.

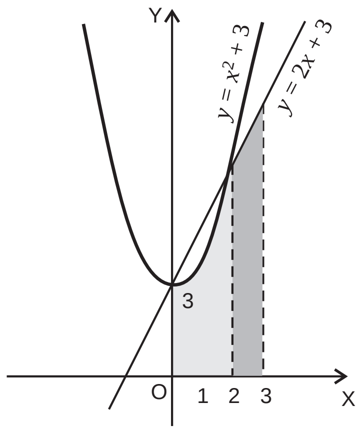


Fig. 8.34.

Solving $y = x^2 + 3$ and $y = 2x + 3$ simultaneously, we get

$$x^2 + 3 = 2x + 3 \Rightarrow x^2 - 2x = 0$$

$$\Rightarrow \quad x(x - 2) = 0 \Rightarrow x = 0, 2.$$

The required area is shown shaded in fig. 8.34. For $0 \le x \le 2$, the area lies below the parabola $y = x^2 + 3$ and for $2 \le x \le 3$, the area lies below the line $y = 2x + 3$.

Hence, the required area

$$= \int_0^2 (x^2 + 3)\,dx + \int_2^3 (2x + 3)\,dx = \left[\frac{x^3}{3} + 3x\right]_0^2 + \left[x^2 + 3x\right]_2^3$$

$$= \left(\frac{8}{3} + 6 - 0\right) + [(9 + 9) - (4 + 6)]$$

$$= \frac{8}{3} + 6 + 8 = \frac{50}{3} \text{ sq. units.}$$

Example 29. *Find the area of the region bounded by $y = x^2 + 1$, $y = x$, $x = 0$ and $y = 2$.*

Solution. First note that the curve $y = x^2 + 1$ *i.e.* $y - 1 = x^2$ represents an upward parabola with vertex at (0, 1) and in the first quadrant :

$$x = \sqrt{y-1}.$$

The required area is shown shaded in fig. 8.35.

$\therefore$ The required area = the area of the shaded region

$$= \int_0^2 y\,dy - \int_1^2 \sqrt{y-1}\,dy$$

$$= \left[\frac{y^2}{2}\right]_0^2 - \left[\frac{(y-1)^{3/2}}{3/2}\right]_1^2$$

$$= (2 - 0) - \frac{2}{3}(1 - 0) = \frac{4}{3} \text{ sq. units.}$$

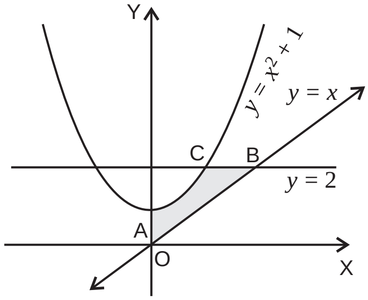


Fig. 8.35.

Example 30. *Find the area of the region* $\left\{(x, y);\ x^2 + y^2 \le 1 \le x + \frac{y}{2}\right\}$.

Solution. The equation $x^2 + y^2 = 1$ ...(*i*)

represents a circle with centre (0, 0) and radius 1 unit.

The equation $x + \frac{y}{2} = 1$ *i.e.* $y = 2 - 2x$ ...(*ii*)

represents a straight line.

The given region $\{(x, y);\ x^2 + y^2 \le 1 \le x + \frac{y}{2}\}$ is bounded by the circle (*i*) and the line (*ii*).

It is shown shaded in fig. 8.36.

The circle and line meet where $x^2 + (2 - 2x)^2 = 1$

$\Rightarrow \quad 5x^2 - 4x + 3 = 0 \Rightarrow x = 1, \frac{3}{5}$.

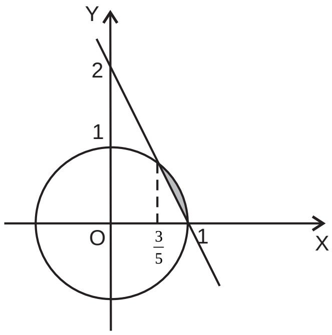


Fig. 8.36.

In the first quadrant, from (*i*), $y = \sqrt{1 - x^2}$.

Hence, the required area

$$= \int_{3/5}^{1} \left(\sqrt{1-x^2} - (2-2x)\right) dx = \left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x - 2x + x^2\right]_{3/5}^{1}$$

$$= \left(0 + \frac{1}{2}\sin^{-1}1 - 2 + 1\right) - \left(\frac{1}{2}\cdot\frac{3}{5}\cdot\frac{4}{5} + \frac{1}{2}\sin^{-1}\frac{3}{5} - 2\cdot\frac{3}{5} + \frac{9}{25}\right)$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} - 1 - \frac{6}{25} - \frac{1}{2}\sin^{-1}\frac{3}{5} + \frac{6}{5} - \frac{9}{25} = \left(\frac{\pi}{4} - \frac{2}{5} - \frac{1}{2}\sin^{-1}\frac{3}{5}\right) \text{ sq. units.}$$

Example 31. *Sketch the region bounded by the curve $y = \sqrt{5 - x^2}$ and $y = |x - 1|$ and find its area.*

Solution. The equation $y = \sqrt{5 - x^2}$ ...(*i*)

represents the upper half of the circle $x^2 + y^2 = 5$ with centre at origin and radius = $\sqrt{5}$.

Also $y = |x - 1| = \begin{cases} x - 1, & x \ge 1 \\ -(x - 1), & x < 1 \end{cases}$

The point(s) of intersection of (*i*) and $y = x - 1$ are given by

$x^2 + (x - 1)^2 = 5 \;\Rightarrow\; 2x^2 - 2x - 4 = 0$

$\Rightarrow \; x^2 - x - 2 = 0 \;\Rightarrow\; (x + 1)(x - 2) = 0$ but $x \ge 1$

$\Rightarrow$ $x = 2,\ \ y = 2 - 1 = 1.$

$\therefore$ The point of intersection is P(2, 1).

The point(s) of intersection of (*i*) and $y = -x + 1$ are given by

$$x^2 + (-x + 1)^2 = 5 \Rightarrow 2x^2 - 2x - 4 = 0 \Rightarrow x^2 - x - 2 = 0$$

$\Rightarrow$ $(x + 1)(x - 2) = 0$ but $x < 1$

$\Rightarrow$ $x = -1,\ \ y = -(-1) + 1 = 2.$

$\therefore$ The point of intersection is Q(–1, 2).

A rough sketch of the region bounded by the given curves is shown in fig. 8.37.

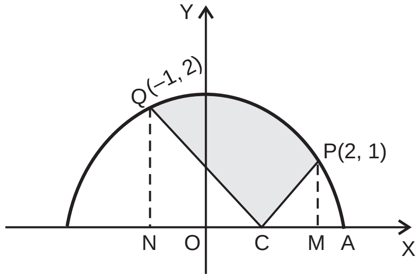


Fig. 8.37.

Hence, the required area = the area of the shaded region

= the area of region QNCMPQ – area of ΔQNC – area of ΔPCM

$$= \int_{-1}^{2} \sqrt{5 - x^2}\, dx - \int_{-1}^{1} (-x + 1)\, dx - \int_{1}^{2} (x - 1)\, dx$$

$$= \left[\frac{x\sqrt{5 - x^2}}{2} + \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}}\right]_{-1}^{2} - \left[\frac{-x^2}{2} + x\right]_{-1}^{1} - \left[\frac{x^2}{2} - x\right]_{1}^{2}$$

$$= \left(1 + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}\right) - \left(-1 + \frac{5}{2}\sin^{-1}\left(\frac{-1}{\sqrt{5}}\right)\right) - \left[\left(-\frac{1}{2} + 1\right) - \left(-\frac{1}{2} - 1\right)\right] - \left[(2 - 2) - \left(\frac{1}{2} - 1\right)\right]$$

$$= \left(\frac{5}{2}\left(\sin^{-1}\frac{2}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{5}}\right) - \frac{1}{2}\right) \text{sq. units.}$$

Example 32. *Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0$, $x = 4$, $y = 0$ and $y = 4$ into three equal parts.* (*C.B.S.E. 2009*)

Solution. The equation $y^2 = 4x$ represents a right hand parabola with vertex (0, 0) and the equation $x^2 = 4y$ represents an upward parabola with vertex (0, 0).

A rough sketch of the two given parabolas and the square bounded by the lines $x = 0$, $x = 4$, $y = 0$ and $y = 4$ is shown in fig. 8.38.

The area bounded between the curves $y^2 = 4x$ and $x^2 = 4y$ is given by

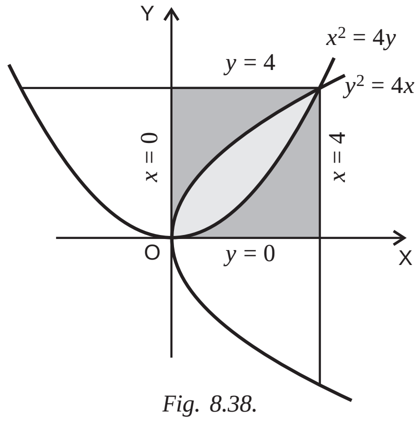


Fig. 8.38.

$$= \int_{0}^{4} \left(2\sqrt{x} - \frac{x^2}{4}\right) = dx$$

$$= \left[2 \cdot \frac{x^{3/2}}{\frac{3}{2}} - \frac{1}{4} \cdot \frac{x^3}{3}\right]_0^4$$

$$= \frac{4}{3}(4^{3/2} - 0) - \frac{1}{12}(4^3 - 0)$$

$$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \text{ sq. units} \qquad \ldots(i)$$

The area of the region bounded by the parabola $x^2 = 4y$ and the lines $x = 4$ and $y = 0$ is given by

$$= \int_{0}^{4} \frac{x^2}{4}\, dx = \frac{1}{4} \cdot \left(\frac{x^3}{3}\right)_0^4 = \frac{16}{3} \text{ sq. units} \qquad \ldots(ii)$$

The area of the region bounded by the parabola $y^2 = 4x$ and the lines $x = 0$ and $y = 4$ is given by

$$= \int_0^4 x\,dy = \int_0^4 \frac{y^2}{4}\,dy = \frac{1}{4}\left[\frac{y^3}{3}\right]_0^4 = \frac{16}{3} \text{ sq. units} \qquad \ldots(iii)$$

From (i), (ii) and (iii), it follows that the area of the square bounded by the given lines is divided by the given curves into three equal parts.

Example 33. *Find the area bounded by the axis of x, the curve $y = 2x^2$ and the tangent to the curve at the point whose abscissa is 2.*

Solution. The given curve is $y = 2x^2$ …(i)

It represents an upward parabola with vertex at (0, 0).

If $x = 2$, then $y = 2 \cdot 2^2 = 8$, therefore, the point on the curve whose abscissa = 2 is P(2, 8).

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = 2 \cdot 2x = 4x.$$

∴ The slope of the tangent to the curve at P = 4 . 2 = 8.

The equation of the tangent to the curve at the point P(2, 8) is

$$y - 8 = 8(x - 2) \qquad | \, y - y_1 = m(x - x_1)$$

$$\Rightarrow \quad y = 8x - 8 \qquad \ldots(ii)$$

It meets the x-axis *i.e.* $y = 0$ at $8x - 8 = 0$ *i.e.* at $x = 1$.

∴ The required area = the area of the shaded region

$$= \int_0^2 2x^2\,dx - \int_1^2 (8x - 8)\,dx = 2 . \left[\frac{x^3}{3}\right]_0^2 - \left[8 . \frac{x^2}{2} - 8x\right]_1^2$$

$$= \frac{2}{3}(8 - 0) - [(16 - 16) - (4 - 8)] = \frac{16}{3} - 4 = \frac{4}{3} \text{ sq. units.}$$

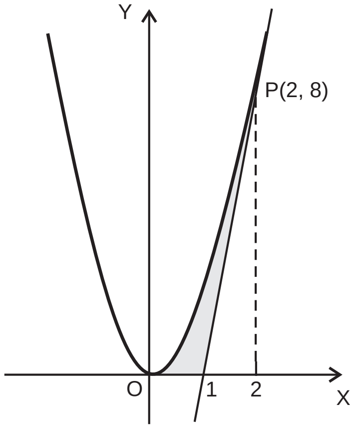


Fig. 8.39.

Example 34. *Using integration, find the area of the region bounded by the line $y - 1 = x$, the x-axis and the ordinates $x = -2$ and $x = 3$.* *(C.B.S.E. 2002 C)*

Solution. The given line is $y - 1 = x$

i.e. $\qquad y = x + 1 \qquad \ldots(i)$

The graph of the line is shown in fig. 8.40. The line meets the x-axis *i.e.* $y = 0$ at the point (–1, 0). Thus, the straight line between the ordinates $x = -2$ and $x = 3$ crosses the x-axis at $x = -1$.

∴ The required area = the area of the shaded region

$$= \int_{-2}^{3} |y|\,dx = \int_{-2}^{3} |x+1|\,dx$$

$$= \int_{-2}^{-1} |x+1|\,dx + \int_{-1}^{3} |x+1|\,dx$$

$$= \int_{-2}^{-1} (-(x+1))\,dx + \int_{-1}^{3} (x+1)\,dx$$

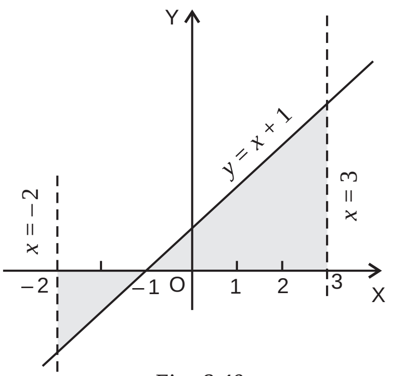


Fig. 8.40.

($\because$ for $-2 \le x \le -1$, $x + 1 \le 0 \Rightarrow |x + 1| = -(x + 1)$; and for $-1 \le x \le 3$, $x + 1 \ge 0 \Rightarrow |x + 1| = x + 1$)

$$= -\left[\frac{x^2}{2}+x\right]_{-2}^{-1} + \left[\frac{x^2}{2}+x\right]_{-1}^{3} = -\left[\left(\frac{1}{2}-1\right)-(2-2)\right]+\left[\left(\frac{9}{2}+3\right)-\left(\frac{1}{2}-1\right)\right]$$

$$= \frac{1}{2} + \frac{9}{2} + 3 + \frac{1}{2} = 8\frac{1}{2} \text{ sq. units.}$$

Example 35. *Find the area of the region bounded by $x = y^3$, the y-axis and the lines $y = -1$, $y = 2$.*

Solution. A rough sketch of the part of the curve $x = y^3$ is shown in fig. 8.41. Note that the curve crosses the y-axis at origin.

$\therefore$ The required area = the area of the shaded region

$$= \int_{-1}^{2} |x|\, dy = \int_{-1}^{2} |y^3|\, dy$$

$$= \int_{-1}^{0} |y^3|\, dy + \int_{0}^{2} |y^3|\, dy$$

$$= \int_{-1}^{0} (-y^3)\, dy + \int_{0}^{2} y^3\, dy$$

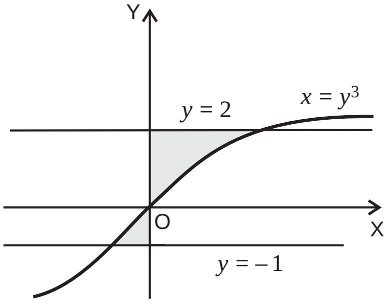


Fig. 8.41.

($\because$ $y^3 \le 0$ for $y \in [-1, 0]$ and $y^3 \ge 0$ for $y \in [0, 2]$)

$$= -\left[\frac{y^4}{4}\right]_{-1}^{0} + \left[\frac{y^4}{4}\right]_{0}^{2} = -\frac{1}{4}(0-1)+\frac{1}{4}(16-0)$$

$$= \frac{1}{4} + 4 = \frac{17}{4} \text{ sq. units.}$$

Example 36. *Find the area enclosed by the curve $y = \cos x$ and the x-axis between $x = 0$ to $x = 2\pi$.* (*NCERT*)

Solution. A rough sketch of the curve $y = \cos x$ between $x = 0$ to $x = 2\pi$ is shown in fig. 8.42. The required area is shown shaded in fig. 8.42. Note that the curve $y = \cos x$ crosses the x-axis at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

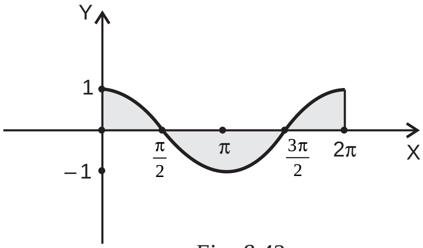


Fig. 8.42.

$\therefore$ The required area $= \int_{0}^{2\pi} |\cos x|\, dx$

$$= \int_{0}^{\pi/2} |\cos x|\, dx + \int_{\pi/2}^{3\pi/2} |\cos x|\, dx + \int_{3\pi/2}^{2\pi} |\cos x|\, dx$$

$$= \int_{0}^{\pi/2} \cos x\, dx + \int_{\pi/2}^{3\pi/2} (-\cos x)\, dx + \int_{3\pi/2}^{2\pi} \cos x\, dx$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{3\pi/2} + [\sin x]_{3\pi/2}^{2\pi}$$

$$= (1 - 0) - [-1 - 1] + [0 - (-1)]$$

$$= 1 + 2 + 1 = 4 \text{ sq. units.}$$

Example 37. *Find the area enclosed between the curves y = sin x and y = cos x that lies between the lines $x = 0$ and $x = \frac{\pi}{2}$.*

Solution. A rough sketch of the curves $y = \sin x$ and $y = \cos x$ between the lines $x = 0$ and $x = \frac{\pi}{2}$ is shown in fig. 8.43.

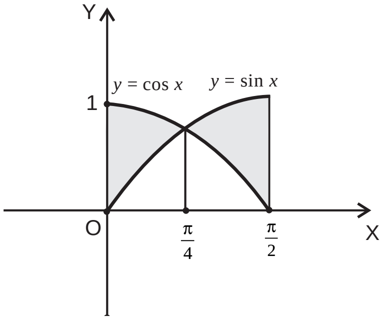


Fig. 8.43.

We note that the curves cross each other at $\sin x = \cos x$ *i.e.* at $\tan x = 1$ *i.e.* at $x = \frac{\pi}{4}$.

The area enclosed between the given curves and the lines $x = 0$ and $x = \frac{\pi}{2}$ is shown shaded in fig. 8.43.

$$\therefore \text{ The required area } = \int_0^{\pi/2} |\cos x - \sin x|\, dx$$

$$= \int_0^{\pi/4} |\cos x - \sin x|\, dx + \int_{\pi/4}^{\pi/2} |\cos x - \sin x|\, dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx$$

$$= \left[\sin x + \cos x\right]_0^{\pi/4} + \left[-\cos x - \sin x\right]_{\pi/4}^{\pi/2}$$

$$= \left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1)\right] - \left[(0+1) - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{2}{\sqrt{2}} - 1 - 1 + \frac{2}{\sqrt{2}} = 2\sqrt{2} - 2 = 2(\sqrt{2} - 1) \text{ sq. units.}$$

EXERCISE 8.1

1. (*i*) Find the area of the region bounded by $y^2 = x$, and the lines $x = 1$, $x = 4$ and the x-axis (in the first quadrant). (*NCERT*)

(*ii*) Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant. (*NCERT*)

(*iii*) Find the area of the region bounded by $y^2 = x - 2$, $x = 4$, $x = 6$ and the x-axis in the first quadrant.

2. (*i*) Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant. (*NCERT*)

(*ii*) Find the area of the region bounded by the curve $ay^2 = x^3$, the y-axis and the lines $y = a$ and $y = 2a$. (*NCERT Examplar Problems*)

3. Using integration, find the area of the region bounded between the line $x = 2$ and the parabola $y^2 = 8x$. (*NCERT Examplar Problems*)

4. Using integration, find the area of the region bounded by the line $2y = 5x + 7$, x-axis and the lines $x = 2$ and $x = 8$. (*NCERT Examplar Problems*)

5. Make a rough sketch of the curve $y = 9 - x^2$, $0 \le x \le 3$ and determine the area enclosed between the curve and the axes.

6. Calculate the area under the curve $y = 2\sqrt{x}$ included between the lines $x = 0$ and $x = 1$. (*NCERT Examplar Problems*)

7. Draw a rough sketch of the curve $y = \sqrt{x-1}$ in the interval [1, 5]. Find the area under the curve and between the lines $x = 1$ and $x = 5$. (*NCERT Examplar Problems*)

8. Find the area bounded by the parabola $x = 4 - y^2$ and the y-axis.

9. Sketch the graph of the curve $y = \sqrt{x} + 1$, $0 \le x \le 4$ and determine the area of the region enclosed by the curve, x-axis and the lines $x = 0$ and $x = 4$.

10. Find the area of the region bounded by the parabola $y^2 = 4ax$ and its latus-rectum. (*NCERT*)

11. (*i*) Find the area lying between the curve $y^2 = 4x$ and the line $y = 2x$. (*NCERT*)

(*ii*) Find the area of the region enclosed by the parabola $y^2 = 4ax$ and the chord $y = mx$. (*NCERT*)

12. Find the area bounded by the curves $y = x$ and $y = x^2$. (*NCERT*)

13. Using definite integrals, find the area of the triangular region whose sides have the equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$. (*C.B.S.E. 2011*)

14. (*i*) Using integration, find the area of the triangle ABC whose vertices have coordinates A (2, 0), B (4, 5) and C (6, 3). (*C.B.S.E. 2006 C, 03 C*)

(*ii*) Using integration, find the area of the triangle ABC, the coordinates of whose vertices are A(4, 1), B(6, 6) and C(8, 4). (*C.B.S.E. 2010*)

(*iii*) Using integration, find the area of the region bounded by the triangle whose vertices are (1, 0), (2, 2) and (3, 1). (*C.B.S.E. 2008*)

15. (*i*) Using integration, find the area bounded by the lines $x + 2y = 2$, $y - x = 1$ and $2x + y = 7$. (*C.B.S.E. 2012, 08*)

(*ii*) Using integration, find the area of the region bounded by the lines $y = 4x + 5$, $y = 5 - x$ and $4y = x + 5$. (*C.B.S.E. 2005*)

16. Find the area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$.

17. Determine the area under the curve $y = \sqrt{r^2 - x^2}$ included between the lines $x = 0$ and $x = r$. Hence, prove that the area of a circle of radius r is πr^2. (*NCERT Examplar Problems*)

18. Sketch the region $\{(x, y) : y = \sqrt{4 - x^2}\}$ and x-axis. Find the area of the region using integration. (*NCERT Examplar Problems*)

19. Sketch the region $\{(x, y) ; 4x^2 + 9y^2 = 36\}$ and find its area, using integration.

20. Make a rough sketch of the curve $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and find

(*i*) the area under the curve and above the x-axis.

(*ii*) the area enclosed by the curve. (*NCERT*)

21. (*i*) Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. (*NCERT*)

(*ii*) Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (*NCERT*)

22. (*i*) Find the area of the region $\{(x, y) : x^2 + y^2 \le 4, x + y \ge 2\}$. (*C.B.S.E. 2012*)

(*ii*) Find the area of the minor segment of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$. (*NCERT*)

23. (*i*) Find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$. (*C.B.S.E. 2004 C*)

(*ii*) Find the area of the region bounded by the curve $y = \sqrt{1 - x^2}$, the line $y = x$ and the positive x-axis. (*C.B.S.E. 2005*)

24. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the straight line $\frac{x}{3}+\frac{y}{2}=1$. (*NCERT*)

25. AOB is a part of the ellipse $9x^2 + y^2 = 36$ in the first quadrant such that OA = 2 and OB = 6. Find the area enclosed by the arc AB and the chord AB. (*NCERT*)

26. Find the area of the region enclosed by the parabola $x^2 = y$ and the straight line $y = x + 2$. (*NCERT*)

27. Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$. (*C.B.S.E. 2013, 06*)

28. Find the area of the region bounded by the parabola $y^2 = 2x$ and the straight line $x - y = 4$. (*NCERT Examplar Problems*)

29. Find the area bounded by the curve $y^2 = 4a^2(x - 1)$ and the lines $x = 1$, $y = 4a$.

30. Draw a rough sketch and find the area of the region bounded by the two parabolas $y^2 = 6x$ and $x^2 = 6y$. (*NCERT Examplar Problems*)

31. Find the area of the region bounded by the parabolas $y^2 = 2px$ and $x^2 = 2py$. (*NCERT Examplar Problems*)

32. Using integration, find the area of the region enclosed between the circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$. (*C.B.S.E. 2013, 08*)

33. Using integration, find the area of the region enclosed between the circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$. (*C.B.S.E. 2008, 07, 06 C*)

34. Find the area of the region $\{(x, y) : y^2 \le 4x,\ 4x^2 + 4y^2 \le 9\}$ using method of integration. (*C.B.S.E. 2013, 08*)

35. Using integration, find the area of the region = $\{(x, y) : y^2 \le 6ax,\ x^2 + y^2 \le 16a^2\}$. (*C.B.S.E. 2013*)

36. Find the area of the smaller region bounded by the curves $y^2 = 2x$ and $x^2 + y^2 = 4x$. (*NCERT Examplar Problems*)

37. Find the area of the region bounded by the parabola $y = x^2 + 1$ and the lines $y = x$, $x = 0$ and $x = 2$.

38. Draw a rough sketch of the curves $y = \sin x$ and $y = \cos x$ as x varies from 0 to $\frac{\pi}{2}$ and find the area of the region enclosed by them and the x-axis.

39. Find the area bounded by the y-axis, $y = \cos x$ and $y = \sin x$ when $0 \le x \le \frac{\pi}{2}$. (*NCERT*)

40. Sketch the graph of $y = |x + 3|$ and evaluate the area under the curve $y = |x + 3|$ above x-axis and between $x = -6$ to $x = 0$. *(C.B.S.E. 2011)*

41. Draw a rough sketch of the curves $y = (x - 1)^2$ and $y = |x - 1|$. Also find the area of the region bounded by these curves.

42. Find the area of the region $\{(x, y) ; 0 \le y \le x^2 + 1,\ 0 \le y \le x + 1,\ 0 \le x \le 2\}$. (*NCERT*)

43. Find the area of the region bounded by the line $y = 3x + 2$, the x-axis and the ordinates $x = -1$ and $x = 1$.

44. Find the area enclosed by the curve $y = \sin x$ and the x-axis between $x = 0$ to $x = 2\pi$. (*NCERT Examplar Problems*)

45. Find the area bounded by the curves $y = x$ and $y = x^3$.

TYPICAL ILLUSTRATIVE EXAMPLES

Example 1. *Find the area bounded by the curve $y = x - 3\sqrt{x}$ and the x-axis.*

Solution. The given curve is $y = x - 3\sqrt{x}$ …(*i*)

The given curve meets the x-axis *i.e.* $y = 0$ at $x - 3\sqrt{x} = 0$

i.e. at $\sqrt{x}(\sqrt{x} - 3) = 0$ *i.e.* at $\sqrt{x} = 0$, $\sqrt{x} = 3$

i.e. at $x = 0$, $x = 9$.

As the portion of the curve between $x = 0$ to $x = 9$ lies below the x-axis, y is negative.

$\therefore$ The area enclosed between the curve and the x-axis

$$= \left|\int_0^9 y\,dx\right| = \left|\int_0^9 (x - 3\sqrt{x})\,dx\right| = \left|\left[\frac{x^2}{2} - 3.\frac{x^{3/2}}{\frac{3}{2}}\right]_0^9\right|$$

$$= \left|\frac{1}{2}[9^2 - 0^2] - 2[9^{3/2} - 0]\right|$$

$$= \left|\frac{1}{2}\times 81 - 2\times 27\right| = \left|-\frac{27}{2}\right| = \frac{27}{2} \text{ sq. units.}$$

Example 2. *Find the area A bounded by the x-axis, part of the curve $y = 1 - \frac{8}{x^2}$ and the ordinates $x = 4$ and $x = 8$.*

If the straight line passing through the point (a, k) and parallel to the y-axis divides the area A into two parts in the ratio 4 : 5, find the value of a.

Solution. Required area A $= \int_4^8 y\,dx = \int_4^8 \left(1 - \frac{8}{x^2}\right) dx = \int_4^8 (1 - 8x^{-2})\,dx$

$$= \left[x - 8.\frac{x^{-1}}{-1}\right]_4^8 = \left[x + \frac{8}{x}\right]_4^8$$

$$= (8 + 1) - (4 + 2) = 3 \text{ sq. units.}$$

The straight line passing through (a, k) and parallel to y-axis is the ordinate $x = a$. It divides the area A into two parts.

Area of 1st part $= \int_4^a \left(1 - \frac{8}{x^2}\right) dx = \left[x + \frac{8}{x}\right]_4^a = a + \frac{8}{a} - 6$ and

area of 2nd part $= \int_a^8 \left(1 - \frac{8}{x^2}\right) dx = \left[x + \frac{8}{x}\right]_a^8 = 9 - \left(a + \frac{8}{a}\right)$.

According to given, $\dfrac{a + \frac{8}{a} - 6}{9 - a - \frac{8}{a}} = \dfrac{4}{5}$

$\Rightarrow \quad \dfrac{a^2 - 6a + 8}{9a - a^2 - 8} = \dfrac{4}{5} \quad \Rightarrow \quad 5a^2 - 30a + 40 = 36a - 4a^2 - 32$

$\Rightarrow \quad 9a^2 - 66a + 72 = 0 \quad \Rightarrow \quad 3a^2 - 22a + 24 = 0$

$\Rightarrow \quad (a - 6)(3a - 4) = 0 \quad \Rightarrow \quad a = 6, \frac{4}{3}$ but $a > 4$

$\Rightarrow \quad a = 6$.

Example 3. *Find the area of the region bounded by the curve $y = x(4 - x)$, the x-axis and the lines $x = 0$, $x = 5$.*

Solution. The given curve is $y = x(4 - x)$.

It can be written as $x^2 - 4x = -y \Rightarrow (x - 2)^2 = -(y - 4)$, which represents a downward parabola with vertex at A(2, 4).

The parabola meets the x-axis *i.e.* $y = 0$ at $4x - x^2 = 0$ *i.e.* at $x = 0$, $x = 4$. A rough sketch of the curve is shown in fig. 8.44.

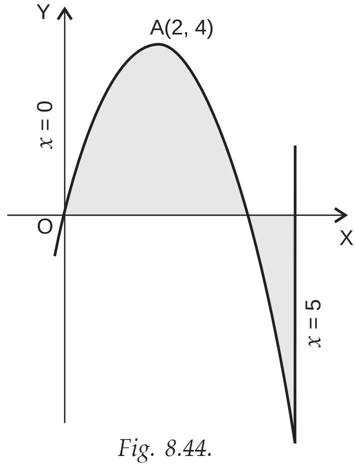


Fig. 8.44.

Note that the curve crosses the x-axis at $x = 0$ and at $x = 4$.

$\therefore$ The required area = the area of shaded region

$$= \int_0^5 |y|\,dx = \int_0^4 |y|\,dx + \int_4^5 |y|\,dx$$

$$= \int_0^4 x(4-x)\,dx + \int_4^5 -x(4-x)\,dx \qquad (\because y \geq 0 \text{ for } x \in [0, 4] \text{ and } y \leq 0 \text{ for } x \in [4, 5])$$

$$= \int_0^4 (4x - x^2)\,dx + \int_4^5 (x^2 - 4x)\,dx = \left[2x^2 - \frac{x^3}{3}\right]_0^4 + \left[\frac{x^3}{3} - 2x^2\right]_4^5$$

$$= \left(32 - \frac{64}{3}\right) - 0 + \left(\frac{125}{3} - 50\right) - \left(\frac{64}{3} - 32\right) = \frac{32}{3} - \frac{25}{3} + \frac{32}{3} = \frac{39}{3} = 13 \text{ sq. units.}$$

Example 4. *Find the area bounded by the cure $y = 2x + x^2 - x^3$, the x-axis and the lines $x = -1$ and $x = 1$.*

Solution. The given curve is $y = 2x + x^2 - x^3$.

It can be written as $y = -x(x^2 - x - 2) = -x(x + 1)(x - 2)$

$\Rightarrow \quad y = -(x + 1)\,x\,(x - 2)$.

The curve meets the x-axis *i.e.* $y = 0$ at $-(x + 1)\,x\,(x - 2) = 0$ *i.e.* at $x = -1, 0, 2$.

We note that the curve crosses the x-axis at $x = -1$, $x = 0$ and at $x = 2$.

Scheme for signs of y, using method of intervals :

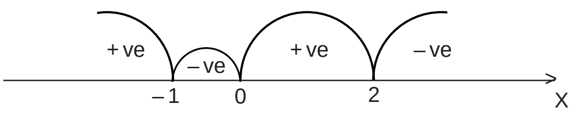


Fig. 8.45.

$\therefore$ The required area $= \displaystyle\int_{-1}^{1} |y|\,dx = \int_{-1}^{0} |y|\,dx + \int_0^1 |y|\,dx$

$$= \int_{-1}^{0} -(2x + x^2 - x^3)dx + \int_0^1 (2x + x^2 - x^3)dx$$

$$= -\left[x^2 + \frac{x^3}{3} - \frac{x^4}{4}\right]_{-1}^{0} + \left[x^2 + \frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= -\left[0 - \left(1 - \frac{1}{3} - \frac{1}{4}\right)\right] + 1 + \frac{1}{3} - \frac{1}{4} = \frac{12 - 4 - 3}{12} + \frac{12 + 4 - 3}{12}$$

$$= \frac{5}{12} + \frac{13}{12} = \frac{18}{12} = \frac{3}{2} \text{ sq. units.}$$

Example 5. *Find the area bounded by the curves $y = 6x - x^2$ and $y = x^2 - 2x$.*

Solution. The given curves are

$$y = 6x - x^2 \quad \ldots(i)$$

and $$y = x^2 - 2x \quad \ldots(ii)$$

The equation (i) can be written as

$$x^2 - 6x + 9 = -y + 9$$

$\Rightarrow$ $(x - 3)^2 = -(y - 9)$, which represents a downward parabola with vertex at (3, 9) and the equation (ii) can be written as

$(x + 1)^2 = y + 1$, which represents an upward parabola with vertex at (1, – 1).

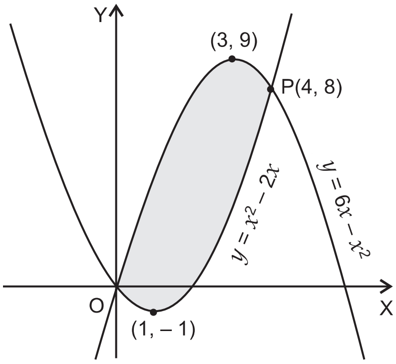


Fig. 8.46.

Solving (i) and (ii), we get

$$6x - x^2 = x^2 - 2x \Rightarrow 2x^2 - 8x = 0$$

$\Rightarrow$ $x(x - 4) = 0 \Rightarrow x = 0, x = 4.$

$\therefore$ The points of intersection are O(0, 0) and P(4, 8).

$\therefore$ The required area = the area of the shaded region

$$= \int_0^4 ((6x - x^2) - (x^2 - 2x))\, dx = \int_0^4 (8x - 2x^2)\, dx$$

$$= \left[8 \cdot \frac{x^2}{2} - 2 \cdot \frac{x^3}{3}\right]_0^4 = \left(64 - \frac{128}{3}\right) - 0 = \frac{64}{3} \text{ sq. units.}$$

EXERCISE 8.2 TYPICAL PROBLEMS

1. Find the area of the region bounded by the curve $xy - 3x - 2y - 10 = 0$, the x-axis and the lines $x = 3$, $x = 4$.

2. Calculate the area of the region bounded by the curve $y = \log x$, the straight line $x = 2$ and the x-axis.

   Hint. The curve $y = \log x$ meets the x-axis *i.e.* $y = 0$ where $\log x = 0$ *i.e.* where $x = 1$.

3. Show that the area included between the x-axis and the curve $a^2 y = x^2 (x + a)$ is $\frac{a^2}{12}$.

   Hint. The curve meets x-axis *i.e.* $y = 0$ where $x^2 (x + a) = 0$ *i.e.* $x = -a, 0$.

4. Find the area bounded by the x-axis, part of the curve $y = 1 + \frac{8}{x^2}$ and the ordinates at $x = 2$ and $x = 4$. If the ordinate at $x = a$ divides the area into two equal parts, find a.

5. Find the area of the region enclosed by the curves $y = \log x$, $y = 2^x$ and the lines $x = \frac{1}{2}$, $x = 2$.

   Hint. Required area $= \int_{1/2}^{2} (2^x - \log x)dx$.

6. Find the area bounded by the curves $y^2 = 4a(x + a)$ and $y^2 = 4b(b - x)$.

   Hint. The equation $y^2 = 4a(x + a)$ represents a right hand parabola with vertex at $(-a, 0)$ and the equation $y^2 = 4b(b - x)$ *i.e.* $y^2 = -4b(x - b)$ represents a left hand parabola with vertex $(b, 0)$. It will be found that their points of intersection are $(b - a, 2\sqrt{ab})$ and $(b - a, -2\sqrt{ab})$. Required area $= 2\int_0^{2\sqrt{ab}} \left(\left(b - \frac{y^2}{4a}\right) - \left(\frac{y^2}{4b} - a\right)\right) dy$.

CHAPTER TEST

1. Find the area under the given curves and the given lines :
(*i*) $y = x^2$, $x = 1$, $x = 2$ and x-axis.
(*ii*) $y = x^4$, $x = 1$, $x = 5$ and x-axis. (*NCERT*)

2. Find the area of the region lying in the first quadrant and bounded by $y = 4x^2$, $x = 0$, $y = 1$ and $y = 4$. (*NCERT*)

3. Find the area enclosed between the curve $y = 2x - x^2$ and the x-axis.

4. Find the area of the region bounded by the parabola $y^2 = 16x$ and its latus-rectum.

5. Find the smaller area cut from the circle $x^2 + y^2 = 25$ by the line $x = 3$.

6. Using integration, find the area of the region bounded by the triangle whose vertices are $(-1, 0)$, $(1, 3)$ and $(3, 2)$. (*NCERT*)

7. Find the area bounded by the parabola $y^2 = 4x$ and the line $y = 2x - 4$.

8. Find the area bounded by the curve $y^2 = 4a^2(x - 3)$ and the lines $x = 3$, $y = 4a$.

9. Draw a rough sketch and find the area of the region bounded by the two parabolas $y^2 = 8x$ and $x^2 = 8y$ by using method of integration.

10. Using integration, find the area of the region $= \{(x, y) : x^2 + y^2 \le 16,\ x^2 \le 6y\}$. (*C.B.S.E. 2013*)

11. Find the area common to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 4x$.

12. Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$. (*NCERT*)

13. Find the area of the region bounded by $y = x^2 + 2$, $y = x$, $x = 0$ and $y = 3$.

14. Sketch the graph of $y = |x + 1|$. Evaluate $\int_{-4}^{2} |x + 1|\,dx$. What does the value of this integral represent on the graph?

15. Find the ratio in which the area bounded by the curves $y^2 = 12x$ and $x^2 = 12y$ is divided by the line $x = 3$.

16. Find the area enclosed by the curve $y = x^3$, the x-axis and the ordinates $x = -2$ and $x = 1$. (*NCERT*)

17. Find the area bounded by the curve $y = x\,|x|$, the x-axis and the ordinates $x = -1$ and $x = 1$. (*NCERT*)

Hint. $y = -x^2$ if $x \le 0$ and $y = x^2$ if $x \ge 0$.

ANSWERS

EXERCISE 8.1

1. (*i*) $\frac{14}{3}$ sq. units (*ii*) $4(4 - \sqrt{2})$ sq. units (*iii*) $\frac{4}{3}(4 - \sqrt{2})$ sq. units

2. (*i*) $\frac{8}{3}(4 - \sqrt{2})$ sq. units (*ii*) $\frac{3}{5}a^2(2^{5/3} - 1)$ sq. units

3. $\frac{32}{3}$ sq. units **4.** 96 sq. units **5.** 18 sq. units

6. $\frac{4}{3}$ sq. units **7.** $\frac{16}{3}$ sq. units **8.** $\frac{32}{3}$ sq. units

9. $\frac{28}{3}$ sq. units **10.** $\frac{8}{3}a^2$ sq. units

11. (*i*) $\frac{1}{3}$ sq. units (*ii*) $\frac{8a^2}{3m^3}$ sq. units **12.** $\frac{1}{6}$ sq. units

13. 8 sq. units

14. (*i*) 7 sq. units (*ii*) 7 sq. units (*iii*) $\frac{3}{2}$ sq. units

15. (*i*) 6 sq. units (*ii*) $\frac{15}{2}$ sq. units **16.** π sq. units

17. $\frac{\pi r^2}{4}$ **18.** 2π sq. units **19.** 6π sq. units

20. (*i*) 3π sq. units (*ii*) 6π sq. units

21. (*i*) 12π sq. units (*ii*) πab sq. units

22. (*i*) $(\pi - 2)$ sq. units (*ii*) $\frac{a^2}{4}$ $(\pi - 2)$ sq. units

23. (*i*) 4π sq. units (*ii*) $\frac{\pi}{8}$ sq. units

24. $\frac{3}{2}$ $(\pi - 2)$ sq. units **25.** $3(\pi - 2)$ sq. units **26.** $\frac{9}{2}$ sq. units

27. $\frac{9}{8}$ sq. units **28.** 18 sq. units **29.** $\frac{16}{3}$ a sq. units

30. 12 sq. units **31.** $\frac{4}{3}p^2$ sq. units **32.** $\left(\frac{8\pi}{3} - 2\sqrt{3}\right)$ sq. units

33. $\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$ sq. units **34.** $\left(\frac{\sqrt{2}}{6} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)\right)$ sq. units

35. $\frac{4}{3}(\sqrt{3} + 4\pi)\, a^2$ sq. units **36.** $2\left(\pi - \frac{8}{3}\right)$ sq. units **37.** $\frac{8}{3}$ sq. units

38. $(2 - \sqrt{2})$ sq. units **39.** $(\sqrt{2} - 1)$ sq. units **40.** 9 sq. units

41. $\frac{1}{3}$ sq. units **42.** $\frac{23}{6}$ sq. units **43.** $\frac{13}{3}$ sq. units

44. 4 sq. units **45.** $\frac{1}{2}$ sq. units

EXERCISE 8.2

1. $(3 + 16 \log 2)$ sq. units. **2.** $(2 \log 2 - 1)$ sq. units. **4.** 4 sq. units ; $a = 2\sqrt{2}$.

5. $\left(\frac{4 - \sqrt{2}}{\log 2} - \frac{5}{2}\log 2 + \frac{3}{2}\right)$ sq. units. **6.** $\frac{8}{3}(a + b)\sqrt{ab}$ sq. units.

CHAPTER TEST

1. (*i*) $\frac{7}{3}$ sq. units (*ii*) $\frac{3124}{5}$ sq. units **2.** $\frac{7}{3}$ sq. units

3. $\frac{4}{3}$ sq. units **4.** $\frac{128}{3}$ sq. units

5. $\left(\frac{25}{2}\pi - 12 - 25\sin^{-1}\left(\frac{3}{5}\right)\right)$ sq. units **6.** 4 sq. units

7. 9 sq. units **8.** $\frac{16}{3}$ a sq. units **9.** $\frac{64}{3}$ sq. units

10. $\frac{4}{3}(\sqrt{3} + 4\pi)$ sq. units **11.** $\left(\frac{8\pi}{3} - 2\sqrt{3}\right)$ sq. units **12.** $\frac{21}{2}$ sq. units

13. $\frac{23}{6}$ sq. units **14.** 9 sq. units **15.** 15 : 49

16. $\frac{17}{4}$ sq. units **17.** $\frac{2}{3}$ sq. units